

Constructions of invariants for surface-links via link invariants and applications to the Kauffman bracket

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
Abstract

In this paper, we formulate a construction of ideal coset invariants for surface-links in 4-space using invariants for knots and links in 3-space. We apply the construction to the Kauffman bracket polynomial invariant and obtain an invariant for surface-links called the Kauffman bracket ideal coset invariant of surface-links. We also define a series of new invariants $\{\mathbf{K}_{2n-1}(\mathcal{L}) | n = 2, 3, 4, \dots\}$ for surface-links \mathcal{L} defined by skein relations, which are more effective than the Kauffman bracket ideal coset invariant to distinguish given surface-links.

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Key words and phrases: ideal coset invariant; knotted surface; marked graph diagram; surface-link; invariant for surface-links; Kauffman bracket polynomial, skein relation.

1 Introduction

By a *surface-link* (or *knotted surface*) we mean a closed 2-manifold smoothly (or piecewise linearly and locally flatly) embedded in the 4-space \mathbb{R}^4 or S^4 . Two surface-links are said to be *equivalent* if they are ambient isotopic. A *marked graph diagram* (or *ch-diagram*) is a link diagram possibly with some 4-valent vertices equipped with markers: . S. J. Lomonaco, Jr. [21] and K. Yoshikawa [25] introduced

a method of presenting surface-links by marked graph diagrams. Indeed, every surface-link \mathcal{L} is presented by an admissible marked graph diagram D . Moreover, if D is an admissible marked graph diagram presenting a surface-link \mathcal{L} , then we can construct a surface-link $\mathcal{S}(D)$ from D such that $\mathcal{S}(D)$ is equivalent to \mathcal{L} . If \mathcal{L} is an oriented surface-link, then it is presented by an admissible oriented marked graph diagram (see Section 2). Using marked graph diagram presentations of surface-links, some properties and invariants for surface-links have been studied by several researchers up to now. For example, see [1, 4, 5, 7, 13, 14, 17, 18, 19, 23, 25] and therein.

In [19], the author defined a polynomial, denoted by $\ll D \gg$, for marked graph diagrams D by a state-sum model associated with an arbitrary given link invariant for knots and links in 3-space as its state evaluation, which is an invariant for marked graphs in the 3-space \mathbb{R}^3 presented by the diagram D and satisfies a skein

relation (see Section 3). In [5], Y. Joung, J. Kim and the author constructed *ideal coset invariants* for surface-links in 4-space by means of the polynomial $\ll D \gg$ for marked graph diagrams D , and applied the construction to the elementary classical link invariant $[K] := A^{\|K\|-1}$, where A is a variable and $\|K\|$ is the number of components of a classical link K and obtained an ideal coset invariant for unoriented (nonorientable or orientable but not oriented) surface-links.

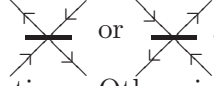
In this paper, we formulate a construction of ideal coset invariants for oriented surface-links in 4-space by means of the polynomial $\ll D \gg$ for oriented marked graph diagrams. When we forget orientation from this formulation, we get a refined construction of ideal coset invariants for unoriented surface-links given in [5] with a simplification that is more applicable in practice. We present a way how to find a unique representative, namely, the *normal form* of a given ideal coset from the polynomial $\ll D \gg$ of an (resp. oriented) marked graph diagram D by using Groebner basis calculation on computer, which is an invariant for the (resp. oriented) surface-link in the 4-space \mathbb{R}^4 or S^4 presented by the (resp. oriented) marked graph diagram D . We apply this construction to the Kauffman bracket polynomial for unoriented knots and links and the normalized Kauffman bracket polynomial for oriented knots and links in 3-space, which lead the *Kauffman bracket ideal coset invariant* for unoriented surface-links and the *normalized Kauffman bracket ideal coset invariant* for oriented surface-links, respectively. Further, by specializing variables of the polynomial $\ll \cdot \gg$ associated with the (resp. normalized) Kauffman bracket polynomial, we define a series of new invariants $\{\mathbf{K}_{2n-1}(\mathcal{L}) | n = 2, 3, 4, \dots\}$ for (resp. oriented) surface-links \mathcal{L} , which are more powerful than the (resp. normalized) Kauffman bracket ideal coset invariant to distinguish given (resp. oriented) surface-links. We also discuss various examples.

This paper is organized as follows. In Section 2, we review (oriented) marked graph diagram presentation of (oriented) surface-links. In Section 3, we deal with the polynomial invariants $\ll \cdot \gg$ for (resp. oriented) marked graphs in the 3-space \mathbb{R}^3 associated with classical (resp. oriented) link invariants. In Section 4, we construct ideal coset invariants for (oriented) surface-links and present how to find the normal form of a given ideal coset in terms of the polynomial $\ll \cdot \gg$ by using the Groebner basis theory. In Section 5, we apply the construction in Section 4 to the (resp. normalized) Kauffman bracket polynomial and derive the (resp. normalized) Kauffman bracket ideal coset invariant for (resp. oriented) surface-links. In Section 6, we define a series of new invariants $\{\mathbf{K}_{2n-1}(\mathcal{L}) | n = 2, 3, 4, \dots\}$ for surface-links \mathcal{L} using skein relations and calculate the invariants for various surface-links in 4-space.

2 Marked graph diagrams of surface-links

In this section, we review marked graph diagrams presenting surface-links. A *marked vertex graph* or simply a *marked graph* is a spatial graph G in \mathbb{R}^3 which satisfies that G is a finite regular graph possibly with 4-valent vertices, say v_1, v_2, \dots, v_n ; each vertex v_i is a rigid vertex (that is, we fix a rectangular neighborhood N_i homeomorphic to $\{(x, y) | -1 \leq x, y \leq 1\}$, where v_i corresponds to the origin and the edges incident to v_i are represented by $x^2 = y^2$); each vertex v_i has a *marker* which is the interval on N_i given by $\{(x, 0) | -\frac{1}{2} \leq x \leq \frac{1}{2}\}$.

An *orientation* of a marked graph G is a choice of an orientation for each edge

of G in such a way that every vertex in G looks like . A marked graph is said to be *orientable* if it admits an orientation. Otherwise, it is said to be *nonorientable*. By an *oriented marked graph* we mean an orientable marked graph with a fixed orientation. Two oriented marked graphs are *equivalent* if they are ambient isotopic in \mathbb{R}^3 with keeping rectangular neighborhoods, orientation and markers. An oriented marked graph G in \mathbb{R}^3 can be described as usual by a diagram D in \mathbb{R}^2 , which is an oriented link diagram in \mathbb{R}^2 possibly with some marked 4-valent vertices whose incident four edges have orientations illustrated as above, and is called an *oriented marked graph diagram* of G (cf. Figure 1).

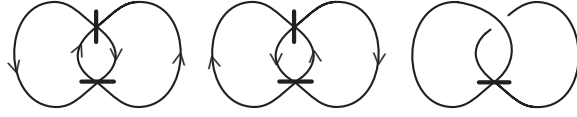


Figure 1: Oriented marked graph diagrams and a nonorientable marked graph diagram

Two oriented marked graph diagrams in \mathbb{R}^2 represent equivalent oriented marked graphs in \mathbb{R}^3 if and only if they are transformed into each other by a finite sequence of oriented mark preserving rigid vertex 4-regular spatial graph moves (simply, *oriented mark preserving RV4 moves*) $\Gamma_1, \Gamma'_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma'_4$ and Γ_5 shown in Figure 2 ([9, 14]), which consist of Yoshikawa moves of type I (see Theorem 2.1).

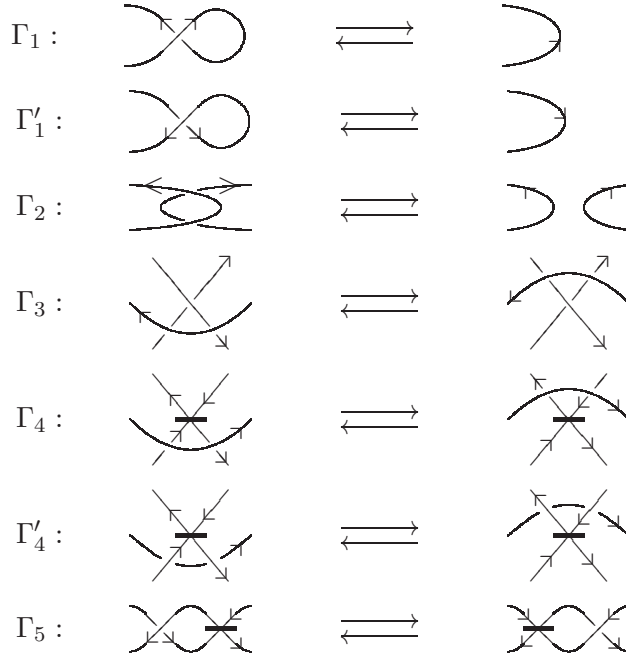


Figure 2: Oriented mark preserving RV4 moves

An *unoriented* marked graph diagram or, simply, a marked graph diagram is a nonorientable or an orientable but not oriented marked graph diagram in \mathbb{R}^2 . Two marked graph diagrams in \mathbb{R}^2 represent equivalent marked graphs in \mathbb{R}^3 if

and only if they are transformed into each other by a finite sequence of the moves $\Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega'_4$ and Ω_5 , where Ω_i stands for the move Γ_i forgetting orientation.

For an (oriented) marked graph diagram D , let $L_-(D)$ and $L_+(D)$ be the (oriented) link diagrams obtained from D by replacing each marked vertex \times with \bigcap (and \bigcup , respectively, as illustrated in Figure 3. We call $L_-(D)$ and $L_+(D)$ the *negative resolution* and the *positive resolution* of D , respectively. An (oriented) marked graph diagram D is *admissible* if both resolutions $L_-(D)$ and $L_+(D)$ are trivial link diagrams.

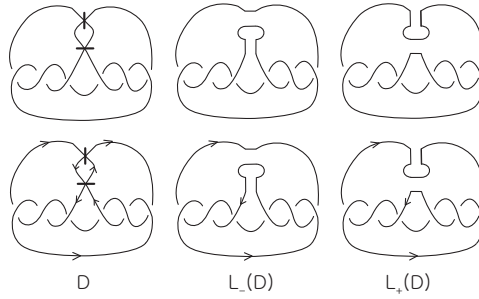


Figure 3: Marked graph diagrams and their resolutions

For $t \in \mathbb{R}$, we denote by \mathbb{R}_t^3 the hyperplane of \mathbb{R}^4 whose fourth coordinate is equal to $t \in \mathbb{R}$, i.e., $\mathbb{R}_t^3 := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_4 = t\}$. A surface-link $\mathcal{L} \subset \mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$ can be described in terms of its *cross-sections* $\mathcal{L}_t = \mathcal{L} \cap \mathbb{R}_t^3$, $t \in \mathbb{R}$ (cf. [3]). Let $p : \mathbb{R}^4 \rightarrow \mathbb{R}$ be the projection given by $p(x_1, x_2, x_3, x_4) = x_4$. Note that any surface-link can be perturbed to a surface-link \mathcal{L} such that the projection $p : \mathcal{L} \rightarrow \mathbb{R}$ is a Morse function with finitely many distinct non-degenerate critical values. More especially, it is well known (cf. [6, 12, 16, 21]) that any surface-link \mathcal{L} can be deformed into a surface-link \mathcal{L}' , called a *hyperbolic splitting* of \mathcal{L} , by an ambient isotopy of \mathbb{R}^4 in such a way that the projection $p : \mathcal{L}' \rightarrow \mathbb{R}$ satisfies that all critical points are non-degenerate, all the index 0 critical points (minimal points) are in \mathbb{R}_{-1}^3 , all the index 1 critical points (saddle points) are in \mathbb{R}_0^3 , and all the index 2 critical points (maximal points) are in \mathbb{R}_1^3 .

Let \mathcal{L} be a surface-link and let \mathcal{L}' be a hyperbolic splitting of \mathcal{L} . Then the cross-section $\mathcal{L}'_0 = \mathcal{L}' \cap \mathbb{R}_0^3$ at $t = 0$ is a spatial 4-valent regular graph in \mathbb{R}_0^3 . We give a marker at each 4-valent vertex (saddle point) that indicates how the saddle point opens up above as illustrated in Figure 4. The resulting marked graph G is called a *marked graph presenting* \mathcal{L} . A diagram of a marked graph presenting \mathcal{L} is clearly admissible, and is called a *marked graph diagram* (or *ch-diagram* (cf. [23])) *presenting* \mathcal{L} .

When \mathcal{L} is an oriented surface-link, we choose an orientation for each edge of \mathcal{L}'_0 that coincides with the induced orientation on the boundary of $\mathcal{L}' \cap \mathbb{R}^3 \times (-\infty, 0]$ by the orientation of \mathcal{L}' inherited from the orientation of \mathcal{L} . The resulting oriented marked graph diagram D is called an *oriented marked graph diagram presenting* \mathcal{L} .

Let D be an admissible marked graph diagram with marked vertices v_1, \dots, v_n .

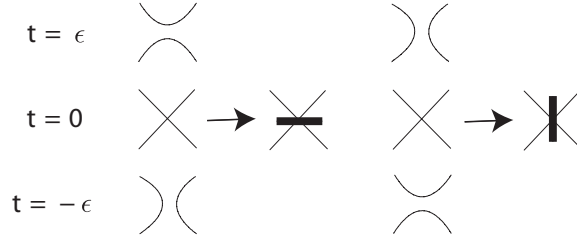
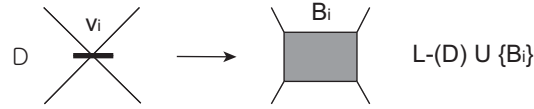


Figure 4: Marking of a vertex


 Figure 5: A band attached to $L_-(D)$ at v_i

We define a surface $F(D) \subset \mathbb{R}^3 \times [-1, 1]$ by

$$(\mathbb{R}^3, F(D) \cap \mathbb{R}_t^3) = \begin{cases} (\mathbb{R}^3, L_+(D)) & \text{for } 0 < t \leq 1, \\ \left(\mathbb{R}^3, L_-(D) \cup \left(\bigcup_{i=1}^n B_i \right) \right) & \text{for } t = 0, \\ (\mathbb{R}^3, L_-(D)) & \text{for } -1 \leq t < 0, \end{cases}$$

where $B_i (i = 1, \dots, n)$ is a band attached to $L_-(D)$ at the marked vertex v_i as shown in Figure 5. We call $F(D)$ the *proper surface associated with D* . Since D is admissible, we can obtain a surface-link from $F(D)$ by attaching trivial disks in $\mathbb{R}^3 \times [1, \infty)$ and another trivial disks in $\mathbb{R}^3 \times (-\infty, 1]$. We denote the resulting surface-link by $\mathcal{S}(D)$, and call it the *surface-link associated with D* . It is known that the isotopy type of $\mathcal{S}(D)$ does not depend on choices of trivial disks (cf. [6, 16]). It is straightforward from the construction of $\mathcal{S}(D)$ that D is a marked graph diagram presenting $\mathcal{S}(D)$. It is known that D is orientable if and only if $F(D)$ is an orientable surface. When D is oriented, the resolutions $L_-(D)$ and $L_+(D)$ have orientations induced from the orientation of D (see Figure 3), and we assume $F(D)$ is oriented so that the induced orientation on $L_+(D) = \partial F(D) \cap \mathbb{R}_1^3$ matches the orientation of $L_+(D)$. Let \mathcal{L} be an oriented surface-link in \mathbb{R}^4 . We say that \mathcal{L} is *presented* by an oriented marked graph diagram D if \mathcal{L} is ambient isotopic to the oriented surface-link $\mathcal{S}(D)$ in \mathbb{R}^4 . Note that any oriented surface-link is presented by an oriented marked graph diagram.

Throughout this paper, a *surface-link* means a nonorientable surface-link or an orientable surface-link without orientation, and an *oriented surface-link* means an orientable surface-link with a fixed orientation. Now we conclude this section by recalling the following:

Theorem 2.1 ([14, 15, 24]). (1) Two oriented marked graph diagrams present the same oriented surface-link if and only if they are transformed into each other by a finite sequence of the oriented mark preserving RV4 moves in Figure 2, called the *oriented Yoshikawa moves of type I*, and the *oriented Yoshikawa moves of type II* in Figure 6.

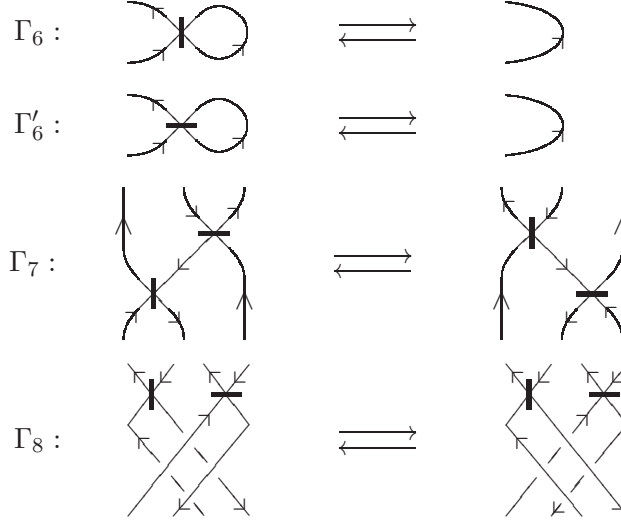


Figure 6: Oriented Yoshikawa moves of type II

(2) Two unoriented marked graph diagrams present the same surface-link if and only if they are transformed into each other by a finite sequence of the unoriented mark preserving RV4 moves $\Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega'_4, \Omega_5$, called the *unoriented Yoshikawa moves of type I*, and the moves $\Omega_6, \Omega'_6, \Omega_7$ and Ω_8 , called the *unoriented Yoshikawa moves of type II*, where Ω_i stands for the move Γ_i without orientation.

3 Polynomial invariants for marked graphs in \mathbb{R}^3 associated with link invariants

In this section, we first review the polynomial invariants for unoriented marked graphs in \mathbb{R}^3 associated with invariants for unoriented knots and links in \mathbb{R}^3 (firstly introduced in [19] and refined in [5]) in a specialized fashion that is more applicable in practice, and then we define polynomial invariants for oriented marked graphs in \mathbb{R}^3 associated with invariants for oriented knots and links in \mathbb{R}^3 .

Let R be a commutative ring with the additive identity 0 and the multiplicative identity 1 and let $[\] : \{ \text{unoriented links in } \mathbb{R}^3 \} \rightarrow R$ be a regular or an ambient isotopy invariant such that for a unit $\alpha \in R$ and an element $\delta \in R$,

$$\left[\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right] = \alpha \left[\begin{array}{c} \diagup \quad \diagup \\ \diagdown \quad \diagdown \end{array} \right], \quad \left[\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right] = \alpha^{-1} \left[\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right], \quad (3.1)$$

$$\left[K \sqcup \bigcirc \right] = \delta \left[K \right], \quad (3.2)$$

where $K \sqcup \bigcirc$ denotes a disjoint union of a circle \bigcirc and a link diagram K .

Let $R[x, y]$ be the ring of polynomials in variables x and y with coefficients in R .

Definition 3.1. For a given marked graph diagram D , let $[[D]](x, y)$ ($[[D]]$ for short) be a polynomial in $R[x, y]$ defined by the following two rules:

(L1) $[[D]] = [D]$ if D is a link diagram,

(L2) $[[\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array}]] = [[\begin{array}{c} \diagup \quad \diagup \\ \diagdown \quad \diagdown \end{array}]]x + [[\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array}]]y.$

Let $D = D_1 \cup \cdots \cup D_m$ be an oriented link diagram and let $w(D_i)$ be the usual writhe of the component D_i . The *self-writhe* $sw(D)$ of D is defined to be the sum $sw(D) = \sum_{i=1}^m w(D_i)$. Now let D be a marked graph diagram. We first choose an arbitrary orientation for each component of $L_+(D)$ and $L_-(D)$. Then we define the *self-writhe* $sw(D)$ of D by

$$sw(D) = \frac{sw(L_+(D)) + sw(L_-(D))}{2}.$$

It is noted that $sw(L_+(D))$ and $sw(L_-(D))$ are independent of the choice of orientations because the writhe of each component of $L_+(D)$ and $L_-(D)$ is independent of the choice of orientation for the component.

Remark 3.2. The self-writhe $sw(D)$ of a marked graph diagram D is invariant under all unoriented Yoshikawa moves except the move Ω_1 . Indeed, it follows from [19, Lemma 4.1] that $t(D) = 2sw(D)$ is invariant under all unoriented Yoshikawa moves and so does $sw(D)$, except the move Ω_1 . For Ω_1 , we have

$$sw\left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array}\right) = sw\left(\begin{array}{c} \diagup \\ \diagdown \end{array}\right) + 1, \quad sw\left(\begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array}\right) = sw\left(\begin{array}{c} \diagdown \\ \diagup \end{array}\right) - 1.$$

Definition 3.3. Let D be a marked graph diagram. We define $\ll D \gg (x, y)$ ($\ll D \gg$ for short) to be the polynomial in $R[x, y]$ given by

$$\ll D \gg = \alpha^{-sw(D)} [[D]](x, y).$$

Let D be a marked graph diagram. A *state* of D is an assignment of T_∞ or T_0 to each marked vertex in D . Let $\mathfrak{S}(D)$ be the set of all states of D . For each state $\sigma \in \mathfrak{S}(D)$, let D_σ denote the link diagram obtained from D by replacing marked vertices of D with trivial 2-tangles according to the assignment T_∞ or T_0 by the state σ :

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \xrightarrow{T_\infty} \begin{array}{c} \diagup \\ \diagdown \end{array} \quad \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} \xrightarrow{T_0} \begin{array}{c} \diagdown \\ \diagup \end{array} \begin{array}{c} \diagup \\ \diagdown \end{array}$$

Then $\ll D \gg$ has the following *state-sum formula*:

$$\ll D \gg = \alpha^{-sw(D)} \sum_{\sigma \in \mathfrak{S}(D)} [D_\sigma] x^{\sigma(\infty)} y^{\sigma(0)}, \quad (3.3)$$

where $\sigma(\infty)$ and $\sigma(0)$ denote the numbers of the assignments T_∞ and T_0 of the state σ , respectively.

Now we define the polynomial $\ll D \gg$ for an oriented marked graph diagram D associated with a given regular or an ambient isotopy invariant

$$[\] : \{\text{oriented links in } \mathbb{R}^3\} \rightarrow R$$

satisfying the properties (3.1) and (3.2) with all possible orientations.

Definition 3.4. For a given oriented marked graph diagram D , let $[[D]](x, y)$ ($[[D]]$ for short) be a polynomial in $R[x, y]$ defined by the following two rules:

(L1) $[[D]] = [D]$ if D is an oriented link diagram,

(L2) $[[\text{crossing}]] = [[\text{arc}]]x + [[\text{arc}]]y.$

Let D be an oriented marked graph diagram. The *writhe* $w(D)$ of D is defined to be the sum of the signs of all crossings in D given by $\text{sign}(\nearrow \searrow) = 1$ and $\text{sign}(\nwarrow \swarrow) = -1$ analogue to the writhe of an oriented link diagram.

Remark 3.5. It is easy to see from Figures 2 and 6 that the writhe $w(D)$ of an oriented marked graph diagram D is invariant under all oriented Yoshikawa moves except the move Γ_1 and Γ'_1 . For Γ_1 , Γ'_1 and their mirror moves, we have

$$\begin{aligned} w(\nearrow \searrow) &= w(\text{arc}) + 1, & w(\nwarrow \swarrow) &= w(\text{arc}) - 1, \\ w(\nwarrow \swarrow) &= w(\text{arc}) + 1, & w(\nearrow \searrow) &= w(\text{arc}) - 1. \end{aligned} \quad (3.4)$$

Definition 3.6. Let D be an oriented marked graph diagram. We define $\ll D \gg$ ((x, y) ($\ll D \gg$ for short)) to be the polynomial in $R[x, y]$ given by

$$\ll D \gg = \alpha^{-w(D)} [[D]](x, y).$$

Let D be an oriented marked graph diagram. A *state* of D is an assignment of T_∞ or T_0 to each marked vertex in D . Let $\mathfrak{S}(D)$ be the set of all states of D . For each state $\sigma \in \mathfrak{S}(D)$, let D_σ denote the oriented link diagram obtained from D by replacing marked vertices of D with trivial oriented 2-tangles according to the assignment T_∞ or T_0 by the state σ :

$$\begin{array}{ccc} \text{crossing} & \longrightarrow & \text{arc} \\ T_\infty & & T_0 \end{array}$$

Then $\ll D \gg$ has the following *state-sum formula*:

$$\ll D \gg = \alpha^{-w(D)} \sum_{\sigma \in \mathfrak{S}(D)} [D_\sigma] x^{\sigma(\infty)} y^{\sigma(0)}. \quad (3.5)$$

Theorem 3.7. Let G be an (resp. oriented) marked graph in \mathbb{R}^3 and let D be an (resp. oriented) marked graph diagram of G . For any given regular or ambient isotopy invariant $[\] : \{(\text{resp. oriented}) \text{ links in } \mathbb{R}^3\} \longrightarrow R$ satisfying the properties (3.1) and (3.2), the polynomial $\ll D \gg$ is invariant under (resp. oriented) Yoshikawa moves of type I in Figure 2. Therefore $\ll D \gg$ is an ambient isotopy invariant of the (resp. oriented) marked graph G in \mathbb{R}^3 .

Proof. Let D be a marked graph diagram of G . It is easy to see that $\ll D \gg (x, y) = \langle \langle D \rangle \rangle (x, 0, 0, y)$ is an invariant for all unoriented Yoshikawa moves of type I (see [5, Lemma 3.3]).

To verify the invariance of $\ll \cdot \gg$ for oriented Yoshikawa moves of type I, we first check the moves Γ_1 and Γ'_1 . It follows from the identities (3.1) with orientations, (3.4) and (3.5) that

$$\ll \nearrow \searrow \gg = \alpha^{-w(\nearrow \searrow)} [[\nearrow \searrow]] = \alpha^{-w(\text{arc})} \alpha^{-1} [[\text{arc}]] = \ll \text{arc} \gg.$$

Similarly, we obtain $\ll \text{diagram} \gg = \ll \text{diagram} \gg$.

Since $[D]$ and the writhe $w(D)$ for an oriented link diagram D are both regular isotopy invariants, it is direct from (3.5) that $\ll \cdot \gg$ is invariant under Γ_2 and Γ_3 . The invariances of $[[\cdot]]$ under the moves Γ_4, Γ'_4 and Γ_5 are seen from Figure 7. Since the writhe $w(D)$ is also invariant under these moves, we see the invariance of $\ll \cdot \gg$ under Γ_4, Γ'_4 and Γ_5 . This completes the proof. \square

$$\begin{aligned}
 [[\text{diagram}]] &= x[[\text{diagram}]] + y[[\text{diagram}]] \\
 &= x[[\text{diagram}]] + y[[\text{diagram}]] = [[\text{diagram}]] \\
 [[\text{diagram}]] &= x[[\text{diagram}]] + y[[\text{diagram}]] \\
 &= x[[\text{diagram}]] + y[[\text{diagram}]] = [[\text{diagram}]] \\
 [[\text{diagram}]] &= x[[\text{diagram}]] + y[[\text{diagram}]] \\
 &= x[[\text{diagram}]] + y[[\text{diagram}]] = [[\text{diagram}]]
 \end{aligned}$$

Figure 7: Invariance of $[[\cdot]]$ under the moves Γ_4, Γ'_4 and Γ_5

4 Ideal coset invariants for surface-links

In this section, we formulate a construction of ideal coset invariants for oriented surface-links by means of the polynomial invariants $\ll \cdot \gg$ for oriented marked graphs in 3-space associated with oriented link invariants. When we forget orientations, this formulation also gives a refinement of the construction of ideal coset invariants for surface-links given in [5] with a simplification that is more applicable in practice. We describe a way how to find a unique representative of an ideal coset in terms of the polynomial $\ll D \gg$ of a marked graph diagram D , which is an invariant of the surface-link presented by D .

4.1 Construction of ideal coset invariants

An *oriented n -tangle diagram* ($n \geq 1$) is an oriented link diagram \mathcal{T} in the rectangle $I^2 = [0, 1] \times [0, 1]$ in \mathbb{R}^2 such that \mathcal{T} transversely intersect with $(0, 1) \times \{0\}$ and $(0, 1) \times \{1\}$ in n distinct points, respectively.

Let $\mathcal{T}_3^{\text{ori}}$ and $\mathcal{T}_4^{\text{ori}}$ denote the set of all oriented 3- and 4-tangle diagrams such that the orientations of the arcs of the tangles intersecting the boundary of I^2 coincide

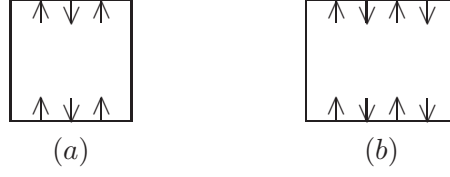
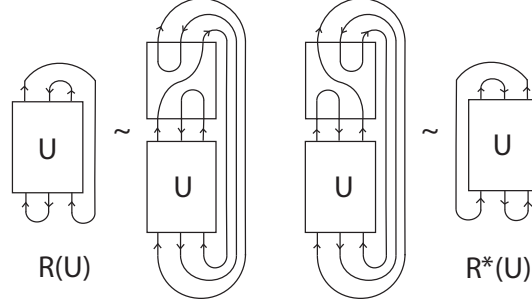
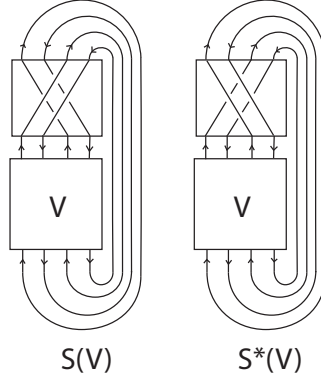


Figure 8: Orientations of arcs in 3, 4-tangle diagrams

Figure 9: Closing operations R and R^* of a 3-tangle U Figure 10: Closing operations S and S^* of a 4-tangle V

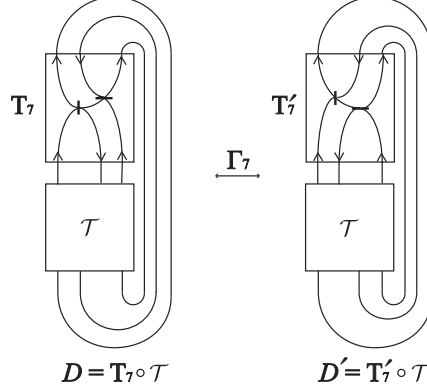
with the orientations as shown in (a) and (b) of Figure 8, respectively. For $U \in \mathcal{T}_3^{\text{ori}}$ and $V \in \mathcal{T}_4^{\text{ori}}$, let $R(U)$, $R^*(U)$, $S(V)$ and $S^*(V)$ denote the oriented link diagrams obtained from the tangles U and V by *closing* n arcs as shown in Figures 9 and 10.

Let \mathcal{T}_3 and \mathcal{T}_4 denote the set of all 3- and 4-tangle diagrams without orientations, respectively. For $U \in \mathcal{T}_3$ and $V \in \mathcal{T}_4$, let $R(U)$, $R^*(U)$, $S(V)$ and $S^*(V)$ be the link diagrams obtained by the same way as above forgetting orientations. Then we have the following three lemmas:

Lemma 4.1. For the oriented Yoshikawa moves Γ_6 and Γ'_6 , we have

$$\llcorner \text{Yoshikawa move } \Gamma_6 \llcorner = (\delta x + y) \llcorner \text{Yoshikawa move } \Gamma'_6 \llcorner, \quad (4.6)$$

$$\llcorner \text{Yoshikawa move } \Gamma'_6 \llcorner = (x + \delta y) \llcorner \text{Yoshikawa move } \Gamma_6 \llcorner. \quad (4.7)$$


 Figure 11: Oriented Yoshikawa move Γ_7

Proof. It follows from Definition 3.4 together with (3.2) that

$$\begin{aligned} \left[\left[\begin{array}{c} \nearrow \searrow \\ \nearrow \searrow \end{array} \right] \right] &= \left[\left[\begin{array}{c} \nearrow \\ \searrow \end{array} \right] \right] x + \left[\left[\begin{array}{c} \nearrow \\ \searrow \end{array} \right] \right] y = (\delta x + y) \left[\left[\begin{array}{c} \nearrow \\ \searrow \end{array} \right] \right], \\ \left[\left[\begin{array}{c} \nearrow \searrow \\ \nearrow \searrow \end{array} \right] \right] &= \left[\left[\begin{array}{c} \nearrow \\ \searrow \end{array} \right] \right] x + \left[\left[\begin{array}{c} \nearrow \\ \searrow \end{array} \right] \right] y = (x + \delta y) \left[\left[\begin{array}{c} \nearrow \\ \searrow \end{array} \right] \right]. \end{aligned}$$

This observation and the fact that the writhe $w(D)$ of an oriented marked graph diagram D is invariant under Γ_6 and Γ'_6 (cf. Remark 3.5) yields that

$$\begin{aligned} \ll \left[\begin{array}{c} \nearrow \searrow \\ \nearrow \searrow \end{array} \right] \gg &= \alpha^{-w(\left[\begin{array}{c} \nearrow \searrow \\ \nearrow \searrow \end{array} \right])} \left[\left[\begin{array}{c} \nearrow \searrow \\ \nearrow \searrow \end{array} \right] \right] \\ &= \alpha^{-w(\left[\begin{array}{c} \nearrow \\ \searrow \end{array} \right])} (\delta x + y) \left[\left[\begin{array}{c} \nearrow \\ \searrow \end{array} \right] \right] = (\delta x + y) \ll \left[\begin{array}{c} \nearrow \\ \searrow \end{array} \right] \gg, \\ \ll \left[\begin{array}{c} \nearrow \searrow \\ \nearrow \searrow \end{array} \right] \gg &= \alpha^{-w(\left[\begin{array}{c} \nearrow \searrow \\ \nearrow \searrow \end{array} \right])} \left[\left[\begin{array}{c} \nearrow \searrow \\ \nearrow \searrow \end{array} \right] \right] \\ &= \alpha^{-w(\left[\begin{array}{c} \nearrow \\ \searrow \end{array} \right])} (x + \delta y) \left[\left[\begin{array}{c} \nearrow \\ \searrow \end{array} \right] \right] = (x + \delta y) \ll \left[\begin{array}{c} \nearrow \\ \searrow \end{array} \right] \gg. \end{aligned}$$

This completes the proof. \square

Lemma 4.2. Let D be an oriented marked graph diagram and let D' be an oriented marked graph diagram obtained from D by applying a single oriented Yoshikawa move Γ_7 . Then

$$\ll D' \gg - \ll D \gg = \alpha^{-w(D)} \sum_{k=1}^m \psi_k(x, y) \left([R(U_k)] - [R^*(U_k)] \right) xy,$$

where $U_k \in \mathcal{T}_3^{ori}$ ($k = 1, 2, \dots, m$) and $\psi_k(x, y)$ is a polynomial in $R[x, y]$.

Proof. Let D be an oriented marked graph diagram and let D' be an oriented marked graph diagram obtained from D by applying a single oriented Yoshikawa move Γ_7 as shown in Figure 11. Applying the axioms (L1) and (L2) in Definition 3.4 to the 3-tangle diagram \mathcal{T} in $D = T_7 \circ \mathcal{T}$, we can express $\ll D \gg$ as a linear combination of

polynomials $[[T_7 \circ U_k]]$ ($1 \leq k \leq m$) for some integer $m \geq 1$, where each U_k is an oriented 3-tangle diagram that has no marked vertices and hence $U_k \in \mathcal{T}_3^{ori}$. Hence

$$[[D]] = [[T_7 \circ \mathcal{T}]] = \sum_{k=1}^m \psi_k(x, y) [[T_7 \circ U_k]],$$

where $\psi_k(x, y)$ is a polynomial in $R[x, y]$. By applying the same procedure to \mathcal{T} in $D' = T'_7 \circ \mathcal{T}$, we have

$$[[D']] = [[T'_7 \circ \mathcal{T}]] = \sum_{k=1}^m \psi_k(x, y) [[T'_7 \circ U_k]].$$

By a straightforward computation, we obtain

$$\begin{aligned} [[T_7 \circ U_k]] &= [\vec{f}_2 \circ U_k]x^2 + [\vec{f}_0 \circ U_k]xy + [\vec{f}_4 \circ U_k]yx + [\vec{f}_1 \circ U_k]y^2, \\ [[T'_7 \circ U_k]] &= [\vec{f}_2 \circ U_k]x^2 + [\vec{f}_0 \circ U_k]xy + [\vec{f}_3 \circ U_k]yx + [\vec{f}_1 \circ U_k]y^2, \end{aligned}$$

where $\vec{f}_0, \dots, \vec{f}_4$ are the fundamental oriented 3-tangle diagrams shown in Figure 12.

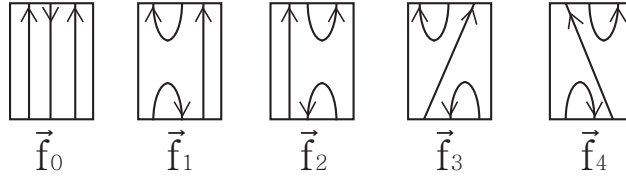


Figure 12: Fundamental oriented 3-tangle diagrams

This gives

$$\begin{aligned} [[D]] - [[D']] &= \sum_{k=1}^m \psi_k(x, y) \left([[T_7 \circ U_k]] - [[T'_7 \circ U_k]] \right) \\ &= \sum_{k=1}^m \psi_k(x, y) \left([\vec{f}_4 \circ U_k] - [\vec{f}_3 \circ U_k] \right) xy \\ &= \sum_{k=1}^m (-\psi_k(x, y)) \left([R(U_k)] - [R^*(U_k)] \right) xy. \end{aligned}$$

Therefore

$$\begin{aligned} \ll D' \gg - \ll D \gg &= \alpha^{-w(D')} [[D']] - \alpha^{-w(D)} [[D]] = \alpha^{-w(D)} \left([[D']] - [[D]] \right) \\ &= \alpha^{-w(D)} \sum_{k=1}^m \psi_k(x, y) \left([R(U_k)] - [R^*(U_k)] \right) xy. \end{aligned}$$

This completes the proof. \square

Lemma 4.3. Let D be an oriented marked graph diagram and let D' be an oriented marked graph diagram obtained from D by applying a single oriented Yoshikawa move Γ_8 . Then

$$\ll D \gg - \ll D' \gg = \alpha^{-w(D)} \sum_{k=1}^n \varphi_k(x, y) ([S(V_k)] - [S^*(V_k)]) xy,$$

where $V_k \in \mathcal{T}_4^{ori}$ ($k = 1, 2, \dots, n$) and $\varphi_k(x, y)$ is a polynomial in $R[x, y]$.

Proof. Let D' be an oriented marked graph diagram obtained from D by applying a single oriented Yoshikawa move Γ_8 as shown in Figure 13. Applying the axioms

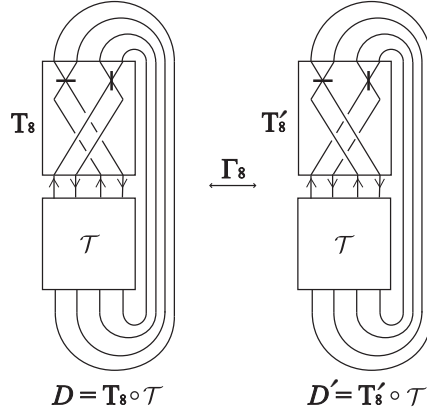


Figure 13: Oriented Yoshikawa move Γ_8

(L1) and (L2) in Definition 3.4 to the 4-tangle diagram \mathcal{T} in $D = T_8 \circ \mathcal{T}$, we can express $[[D]]$ as a linear combination of polynomials $[[T_8 \circ V_k]]$ ($1 \leq k \leq n$) for some integer $n \geq 1$, where each V_k is an oriented 4-tangle diagram that has no marked vertices and hence $V_k \in \mathcal{T}_4^{ori}$. Hence

$$[[D]] = [[T_8 \circ \mathcal{T}]] = \sum_{k=1}^n \varphi_k(x, y) [[T_8 \circ V_k]],$$

where $\varphi_k(x, y)$ is a polynomial in $R[x, y]$. By applying the same procedure to \mathcal{T} in $D' = T'_8 \circ \mathcal{T}$, we have

$$[[D']] = [[T'_8 \circ \mathcal{T}]] = \sum_{k=1}^n \varphi_k(x, y) [[T'_8 \circ V_k]].$$

By a straightforward computation, we obtain

$$\begin{aligned} [[T_8 \circ V_k]] &= [\vec{g}_9 \circ V_k]x^2 + [\vec{g}_5 \circ V_k]xy + [\vec{g} \circ V_k]yx + [\vec{g}_{12} \circ V_k]y^2, \\ [[T'_8 \circ V_k]] &= [\vec{g}_9 \circ V_k]x^2 + [\vec{g}_5 \circ V_k]xy + [\vec{g}^* \circ V_k]yx + [\vec{g}_{12} \circ V_k]y^2, \end{aligned}$$

where $\vec{g}_9, \vec{g}_5, \vec{g}_{12}, \vec{g}$ and \vec{g}^* are the fundamental oriented 4-tangle diagrams shown in Figure 14.

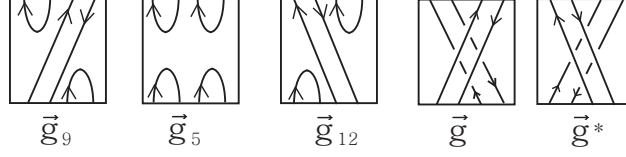


Figure 14: Fundamental oriented 4-tangle diagrams

This gives

$$\begin{aligned}
[[D]] - [[D']] &= \sum_{k=1}^n \varphi_k(x, y) \left([[T_8 \circ V_k]] - [[T'_8 \circ V_k]] \right) \\
&= \sum_{k=1}^n \varphi_k(x, y) \left([\vec{g} \circ V_k] - [\vec{g}^* \circ V_k] \right) xy \\
&= \sum_{k=1}^n \varphi_k(x, y) \left([S(V_k)] - [S^*(V_k)] \right) xy.
\end{aligned}$$

Therefore

$$\begin{aligned}
\ll D \gg - \ll D' \gg &= \alpha^{-w(D)} [[D]] - \alpha^{-w(D')} [[D']] = \alpha^{-w(D)} \left([[D]] - [[D']] \right) \\
&= \alpha^{-w(D)} \sum_{k=1}^n \varphi_k(x, y) \left([S(V_k)] - [S^*(V_k)] \right) xy.
\end{aligned}$$

This completes the proof. \square

Definition 4.4. For any given regular or ambient isotopy invariant

$$[\] : \{(\text{resp. oriented}) \text{ links in } \mathbb{R}^3\} \longrightarrow R$$

satisfying the properties (3.1) and (3.2) (resp. with orientations), the $[\]$ -*obstruction ideal* (simply, $[\]$ *ideal*) or the *ideal associated with* $[\]$ is defined to be the ideal of $R[x, y]$ generated by the polynomials in $R[x, y]$:

$$\begin{aligned}
P_1(x, y) &= \delta x + y - 1, \\
P_2(x, y) &= x + \delta y - 1, \\
P_U(x, y) &= ([R(U)] - [R^*(U)])xy, U \in \mathcal{T}_3 \text{ (resp. } \mathcal{T}_3^{\text{ori}}), \\
P_V(x, y) &= ([S(V)] - [S^*(V)])xy, V \in \mathcal{T}_4 \text{ (resp. } \mathcal{T}_4^{\text{ori}}).
\end{aligned} \tag{4.8}$$

We remark that if $[\]$ is a regular or ambient isotopy invariant for unoriented knots and links in 3-space, then the $[\]$ ideal in Definition 4.4 is identical to the ideal in [5, Definition 4.1] with the specialization given by $(x, y, z, w) = (x, 0, 0, y)$ and $\phi = \text{identity}$.

Theorem 4.5. Let $[\]$ be a regular or ambient isotopy invariant for (resp. oriented) knots and links in the 3-space \mathbb{R}^3 or S^3 and let $I = I([\])$ denote the ideal associated with $[\]$. Then the map

$$\ll \gg^I : \{(\text{resp. oriented}) \text{ marked graph diagrams}\} \longrightarrow R[x, y]/I$$

defined by $\ll \gg^I (D) = \ll D \gg^I := \ll D \gg + I$ for each (resp. oriented) marked graph diagram D is an invariant for (resp. oriented) surface-links in the 4-space \mathbb{R}^4 or S^4 .

Proof. By [5, Theorem 4.2], we see that $\ll \gg^I$ is an invariant for unoriented surface-links and so the oriented case only remain to be proved.

Let D be an oriented marked graph diagram. By Theorem 3.7, we see that $\ll D \gg$ is invariant under the oriented Yoshikawa moves $\Gamma_1, \Gamma'_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma'_4$ and Γ_5 and so clearly do the $[\]$ ideal coset $\ll D \gg + I$.

For the moves Γ_6 and Γ'_6 , it is direct from Lemma 4.1 that

$$\ll \begin{array}{c} \nearrow \searrow \\ \times \end{array} \gg + I = \ll \begin{array}{c} \nearrow \\ \searrow \end{array} \gg + I \text{ and } \ll \begin{array}{c} \nwarrow \swarrow \\ \times \end{array} \gg + I = \ll \begin{array}{c} \nwarrow \\ \swarrow \end{array} \gg + I.$$

Let D' be an oriented marked graph diagram obtained from D by applying a single oriented Yoshikawa move Γ_7 or Γ_8 . Then it is direct from Lemma 4.2 and Lemma 4.3 that $\ll D' \gg - \ll D \gg \in I$. This gives $\ll D \gg + I = \ll D' \gg + I$ and completes the proof. \square

Let \mathbb{F}_R be an extension field of R . By Hilbert Basis Theorem (cf. [2]), the $[\]$ ideal I in $\mathbb{F}_R[x, y]$ is completely determined by a finite number of polynomials in $\mathbb{F}_R[x, y]$, say p_1, p_2, \dots, p_r . Then $I = \langle p_1, p_2, \dots, p_r \rangle$ in $\mathbb{F}_R[x, y]$.

The following corollary is an immediate consequence of Theorem 4.5.

Corollary 4.6. Let S be a commutative ring and let $\phi : R[x, y] \rightarrow S$ be a homomorphism such that $\phi(I) = 0$. Then the composite map

$$\Phi = \phi \circ \ll \gg^I : \{(\text{resp. oriented}) \text{ marked graph diagrams} \} \longrightarrow S$$

defined by $\Phi(D) = \phi(\ll D \gg^I) = \phi(\ll D \gg + I) = \phi(\ll D \gg)$ for each (resp. oriented) marked graph diagram D is an invariant for (resp. oriented) surface-links.

It is noted that a specialization of the polynomial $\ll \gg$ valued in some ring S formalized in Corollary 4.6 is sometimes useful in practice. In the final section 6, we shall discuss such a specialization of the polynomial $\ll \gg$ associated with the (normalized) Kauffman bracket polynomial (see Subsection 5.2) for (oriented) knots and links in 3-space, which gives a series of new invariants for (oriented) surface-links in 4-space.

4.2 Normal forms of ideal cosets

In this subsection, we give a brief description of a way how to find a unique representative of the ideal coset invariant $\ll D \gg + I$ in terms of the polynomial $\ll D \gg$ of a marked graph diagram D by using a Groebner basis for the $[\]$ ideal I , which is indeed an invariant of the surface-link in the 4-space \mathbb{R}^4 or S^4 presented by D .

Let \mathbb{F} be an arbitrary field and let $\mathbb{F}[x_1, \dots, x_n]$ be the ring of polynomials in n variables x_1, \dots, x_n with coefficients in \mathbb{F} . Let I denote the ideal of $\mathbb{F}[x_1, \dots, x_n]$ generated by the polynomials f_1, \dots, f_s and let $G = \{g_1, \dots, g_t\}$ be a Groebner basis for the ideal I with respect to a fixed monomial order. Then it is well known that for any polynomial $f \in \mathbb{F}[x_1, \dots, x_n]$, there exists a unique polynomial r with the following properties:

- (i) No term of r is divisible by any of $\text{LT}(g_1), \dots, \text{LT}(g_t)$, where $\text{LT}(g_i)$ denotes the leading term of f .
- (ii) There exist $b_i \in \mathbb{F}[x_1, \dots, x_n] (1 \leq i \leq t)$ such that

$$f = b_1 g_1 + \dots + b_t g_t + r. \quad (4.9)$$

- (iii) r is the remainder on division of f by $G = \{g_1, \dots, g_t\}$ no matter how the elements of G are listed when using the Division Algorithm in $\mathbb{F}[x_1, \dots, x_n]$.

The unique remainder r in (4.9) is called the *normal form* of f on division by the ideal I and denoted by \bar{f}^G . We should note that if G and G' are Groebner basis for the ideal I with respect to the same monomial order in $\mathbb{F}[x_1, \dots, x_n]$, then $\bar{f}^G = \bar{f}^{G'}$ for all $f \in \mathbb{F}[x_1, \dots, x_n]$ and hence we may denote the unique remainder \bar{f}^G by \bar{f}^I or simply \bar{f} once a monomial order is fixed. For more details, see [5] or [2, Chapter 2].

Now we turn to the ideal coset invariant for surface-links in the subsection 4.1.

Theorem 4.7. Let D be an (resp. oriented) marked graph diagram and let \mathcal{L} be the (resp. oriented) surface-link presented by D . Let $\ll D \gg$ be the polynomial of D defined in Definition 3.3 (resp. Definition 3.6) and let $I = I([\])$ be the ideal associated with a regular or ambient isotopy invariant $[\]$ of (resp. oriented) knots and links in 3-space. Then for any Groebner basis G for the ideal I with a fixed monomial order, the normal form $\overline{\ll D \gg}$ on division of $\ll D \gg$ in \mathbb{F}_R by G is uniquely determined by the surface link \mathcal{L} and therefore it is an invariant of \mathcal{L} . We denote it by $\overline{\ll \mathcal{L} \gg}$.

Proof. This follows directly from Theorem 4.5 and the theory of Groebner bases for ideals in polynomial rings. \square

It is worth noting that by using the commercial computer algebra systems “Maple” or “Mathematica” one can compute the normal form \bar{f} for any polynomial f in some polynomial rings $\mathbb{F}[x_1, \dots, x_n]$ on division by any Groebner basis G for the ideal I with a fixed monomial order such that $f + I = g + I$ if and only if $\bar{f} = \bar{g}$ for all $f, g \in \mathbb{F}[x_1, \dots, x_n]$.

5 Kauffman bracket ideal coset invariants

In this section, we apply the construction formulated in Section 4 to the Kauffman bracket polynomial for knots and links in the 3-space \mathbb{R}^3 or S^3 and compute the ideal coset invariant for unoriented surface-links in the 4-space \mathbb{R}^4 or S^4 associated with the Kauffman bracket polynomial for unoriented link diagrams and the ideal coset invariant for oriented surface-links associated with the normalized Kauffman bracket polynomial for oriented link diagrams. We also describe how to find unique representatives of the ideal cosets $\ll D \gg + I$ in terms of a Groebner basis for the ideal I and the polynomial $\ll D \gg$.

5.1 Kauffman bracket ideal coset invariant for unoriented surface-links

Let L be a link diagram. The Kauffman bracket polynomial of L [8] is a Laurent polynomial $\langle L \rangle = \langle L \rangle(A) \in R = \mathbb{Z}[A^{\pm 1}]$ defined by the following three rules:

$$(B1) \quad \langle \bigcirc \rangle = 1.$$

$$(B2) \quad \langle \bigcirc \sqcup K \rangle = \delta \langle K \rangle, \text{ where } \delta = -A^2 - A^{-2}.$$

$$(B3) \quad \left\langle \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right\rangle = A \left\langle \begin{array}{c} \diagup \\ \diagup \end{array} \right\rangle \left\langle \begin{array}{c} \diagdown \\ \diagdown \end{array} \right\rangle + A^{-1} \left\langle \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} \right\rangle.$$

Note that the Kauffman bracket polynomial is a regular isotopy invariant for unoriented links, namely it is invariant under Reidemeister moves Ω_2 and Ω_3 except the move Ω_1 . For Ω_1 , we have

$$\left\langle \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right\rangle = -A^3 \left\langle \begin{array}{c} \diagup \\ \diagup \end{array} \right\rangle \left\langle \begin{array}{c} \diagdown \\ \diagdown \end{array} \right\rangle, \quad \left\langle \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} \right\rangle = -A^{-3} \left\langle \begin{array}{c} \diagup \\ \diagup \end{array} \right\rangle \left\langle \begin{array}{c} \diagdown \\ \diagdown \end{array} \right\rangle.$$

Now let D be a marked graph diagram. Then it follows from Definitions 3.1 and 3.3 that $\ll D \gg$ is a polynomial in $\mathbb{Z}[A^{\pm 1}][x, y] (= \mathbb{Z}[A^{\pm 1}, x, y])$ given by

$$\ll D \gg = (-A^3)^{-sw(D)} [[D]], \quad (5.10)$$

where $[[D]] = [[D]](A, A^{-1}, x, y)$ is a polynomial defined by the two axioms:

$$(L1) \quad [[D]] = \langle D \rangle \text{ if } D \text{ is a knot or link diagram.}$$

$$(L2) \quad [[\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array}]] = [[\begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array}]]x + [[\begin{array}{c} \diagup \\ \diagup \end{array}]] \left[\begin{array}{c} \diagdown \\ \diagdown \end{array} \right] y.$$

Also, the state-sum formula for the polynomial $\ll D \gg$ in (3.3) is read as

$$\ll D \gg = (-A^3)^{-sw(D)} \sum_{\sigma \in \mathfrak{S}(D)} \langle D_{\sigma} \rangle x^{\sigma(\infty)} y^{\sigma(0)}.$$

Theorem 5.1. (1) The Kauffman bracket ideal I is the ideal of the ring $\mathbb{Z}[A^{\pm 1}, x, y]$ generated by the following three polynomials:

$$\begin{aligned} f_1(A, A^{-1}, x, y) &= (-A^2 - A^{-2})x + y - 1, \\ f_2(A, A^{-1}, x, y) &= x + (-A^2 - A^{-2})y - 1, \\ f_3(A, A^{-1}, x, y) &= (A^4 + 1 + A^{-4})xy. \end{aligned} \quad (5.11)$$

(2) For any marked graph diagram D , the ideal coset $\ll D \gg^I = \ll D \gg + I$ is an invariant for the unoriented surface-link in \mathbb{R}^4 or S^4 presented by D .

Proof. (1) Since $\delta = -A^2 - A^{-2}$, the polynomials $P_1(x, y)$ and $P_2(x, y)$ in (4.8) are the polynomials $f_1(A, A^{-1}, x, y)$ and $f_2(A, A^{-1}, x, y)$ in (5.11), respectively.

Let U be any 3-tangle diagram in \mathcal{T}_3 . Applying the Kauffman bracket axioms (B1)–(B3) to the 3-tangle diagram U in $R(U)$ and $R^*(U)$, we have

$$\langle R(U) \rangle = \sum_{i=0}^5 \psi_i(U) \langle R(f_i) \rangle, \quad \langle R^*(U) \rangle = \sum_{i=0}^5 \psi_i(U) \langle R^*(f_i) \rangle,$$

where $\psi_i(U) \in \mathbb{Z}[A^{\pm 1}, x, y]$ and $f_i (0 \leq i \leq 4)$ denotes the fundamental oriented 3-tangle diagram f_i in Figure 12 without orientation. It is easy to check that

$$\langle R(f_i) \rangle - \langle R^*(f_i) \rangle = \begin{cases} 0, & i = 0, 1, 2; \\ \delta^2 - 1, & i = 3; \\ 1 - \delta^2, & i = 4. \end{cases}$$

From (4.8), we have

$$\begin{aligned} P_U(x, y) &= (\langle R(U) \rangle - \langle R^*(U) \rangle)xy = xy \sum_{i=0}^5 \psi_i(U) (\langle R(f_i) \rangle - \langle R^*(f_i) \rangle) \\ &= (\delta^2 - 1)xy (\psi_3(U) - \psi_4(U)) = (A^4 + 1 + A^{-4})xy (\psi_3(U) - \psi_4(U)). \end{aligned} \quad (5.12)$$

Similarly, let V be any 4-tangle diagram in \mathcal{T}_4 . Applying the Kauffman bracket axioms **(B1)**–**(B3)** to the 4-tangle diagram V in $S(V)$ and $S^*(V)$, we have

$$\langle S(V) \rangle = \sum_{i=0}^{13} \varphi_i(V) \langle S(g_i) \rangle, \quad \langle S^*(V) \rangle = \sum_{i=0}^{13} \varphi_i(V) \langle S^*(g_i) \rangle,$$

where $\varphi_i(V) \in \mathbb{Z}[A^{\pm 1}, x, y]$ and $g_i (0 \leq i \leq 13)$ is the fundamental 4-tangle diagram shown in Figure 15. It is easy but tedious to check (cf. [18, Lemma 4.2]) that

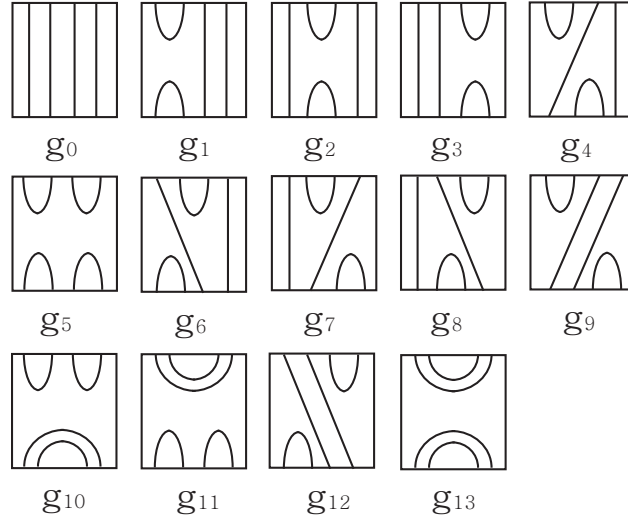


Figure 15: Fundamental 4-tangle diagrams

$$\langle S(g_i) \rangle - \langle S^*(g_i) \rangle = \begin{cases} -xyA^{-10}(A^{20} + A^{12} - A^8 - 1), & i = 0; \\ xyA^{-10}(A^{20} + A^{12} - A^8 - 1), & i = 13; \\ 0 & \text{otherwise.} \end{cases}$$

From (4.8), we have

$$\begin{aligned}
 P_V(x, y) &= (\langle S(V) \rangle - \langle S^*(V) \rangle)xy = xy \sum_{i=0}^{13} \varphi_i(V) (\langle S(g_i) \rangle - \langle S^*(g_i) \rangle) \\
 &= -A^{-10}(A^{20} + A^{12} - A^8 - 1)xy (\varphi_0(V) - \varphi_{13}(V)) \\
 &= -A^{-6}(A^4 + 1 + A^{-4})xy(A^{12} - A^8 + A^4 - 1)(\varphi_0(V) - \varphi_{13}(V)). \quad (5.13)
 \end{aligned}$$

By (5.12) and (5.13), we obtain that for all $U \in \mathcal{T}_3$ and $V \in \mathcal{T}_4$, the polynomials $P_U(x, y)$ and $P_V(x, y)$ are generated by $f_3(A, A^{-1}, x, y) = (A^4 + 1 + A^{-4})xy$. Therefore $I = \langle P_1, P_2, \{P_U, P_V | U \in \mathcal{T}_3, V \in \mathcal{T}_4\} \rangle = \langle f_1, f_2, f_3 \rangle$.

The assertion (2) is direct from Theorem 4.5 by taking $[\]$ to be the Kauffman bracket $\langle \ \rangle$. This completes the proof. \square

5.2 Normalized Kauffman bracket ideal coset invariant for oriented surface-links

Let L be an oriented link diagram and let \tilde{L} be the link diagram L without orientation. The normalized Kauffman bracket polynomial $\langle L \rangle_N$ of L is defined by

$$\langle L \rangle_N = (-A^3)^{-w(L)} \langle \tilde{L} \rangle \quad (5.14)$$

and it is an invariant of the oriented link in \mathbb{R}^3 or S^3 presented by L . It is well known that

$$A^4 \langle \text{crossing} \rangle_N - A^{-4} \langle \text{crossing} \rangle_N = (A^{-2} - A^2) \langle \text{cup} \rangle_N \quad (5.15)$$

and $V_L(t) = \langle L \rangle_N(t^{-1/4})$ is the Jones polynomial of L [8].

Now let D be an oriented marked graph diagram. Since the normalized Kauffman bracket $\langle \ \rangle_N$ is an ambient isotopy invariant, we have $\alpha = 1$. Hence it follows from Definitions 3.4 and 3.6 that the polynomial $\ll D \gg$ associated with $[\] = \langle \ \rangle_N$ is just equal to the polynomial $[[D]] = [[D]](A, A^{-1}, x, y) \in \mathbb{Z}[A^{\pm 1}][x, y] = \mathbb{Z}[A^{\pm 1}, x, y]$ defined by the two axioms:

(L1) $[[D]] = \langle D \rangle_N$ if D is an oriented link diagram,

(L2) $[[\text{crossing}]]$ $= [[\text{cup}]]$ $x + [[\text{cup}]]$ y .

In what follows we denote the polynomial $\ll D \gg$ associated with the normalized bracket $[\] = \langle \ \rangle_N$ by $\ll D \gg_N$ for our convenience. Note that for any state $\sigma \in \mathfrak{S}(D)$, $w(D_\sigma) = w(D)$. From (3.5) and (5.14), we see that the state-sum formula for the polynomial $\ll D \gg_N = [[D]]$ is given by

$$\begin{aligned}
 \ll D \gg_N &= \sum_{\sigma \in \mathfrak{S}(D)} \langle D_\sigma \rangle_N x^{\sigma(\infty)} y^{\sigma(0)} = \sum_{\sigma \in \mathfrak{S}(D)} (-A^3)^{-w(D_\sigma)} \langle \tilde{D}_\sigma \rangle x^{\sigma(\infty)} y^{\sigma(0)} \\
 &= (-A^3)^{-w(D)} \sum_{\sigma \in \mathfrak{S}(D)} \langle \tilde{D}_\sigma \rangle x^{\sigma(\infty)} y^{\sigma(0)}. \quad (5.16)
 \end{aligned}$$

The following theorem 5.2 gives a method of computing the polynomial $\ll D \gg_N$ recursively for a given oriented marked graph diagram D .

Theorem 5.2. Let D be an oriented marked graph diagram.

- (1) $\ll \bigcirc \gg_N = 1$.
- (2) If D and D' are two oriented marked graph diagrams related by a finite sequence of oriented Yoshikawa moves generated by the moves $\Gamma_1, \Gamma'_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma'_4$ and Γ_5 of type I, then $\ll D \gg_N = \ll D' \gg_N$.
- (3) $\ll D \sqcup \bigcirc \gg_N = (-A^{-2} - A^2) \ll D \gg_N$.
- (4) $\ll \begin{array}{c} \nearrow \nwarrow \\ \nwarrow \nearrow \end{array} \gg_N = \ll \begin{array}{c} \frown \\ \smile \end{array} \gg_N x + \ll \begin{array}{c} \rceil \\ \lceil \end{array} \gg_N y$.
- (5) $A^4 \ll \begin{array}{c} \nearrow \nwarrow \\ \nwarrow \nearrow \end{array} \gg_N - A^{-4} \ll \begin{array}{c} \nearrow \nwarrow \\ \nwarrow \nearrow \end{array} \gg_N = (A^{-2} - A^2) \ll \begin{array}{c} \rceil \\ \lceil \end{array} \gg_N$, where D_+, D_- and D_0 are three identical oriented link diagrams except the parts indicated.

Proof. By **(B1)**, **(B2)**, Definition 3.4 and Theorem 3.7, the assertions (1), (2) and (3) follow at once. Since $\ll D \gg_N = [[D]]$ by definition, the skein relation (4) is straightforward from the axiom **(L2)**. Finally, the skein relation in (5) follows immediately from the skein relation in (5.15) for the normalized Kauffman bracket for oriented link diagrams and the axiom **(L1)**. This completes the proof. \square

Theorem 5.3. Let D be an oriented marked graph diagram D and let \tilde{D} be the marked graph diagram D without orientation. Then

$$\ll D \gg_N = (-A^3)^{sw(D)-w(D)} \ll \tilde{D} \gg.$$

Proof. Since $sw(D) = sw(\tilde{D})$ and $\mathfrak{S}(D) = \mathfrak{S}(\tilde{D})$, it follows from (5.16) that

$$\begin{aligned} \ll D \gg_N &= (-A^3)^{-w(D)} \sum_{\sigma \in \mathfrak{S}(D)} \langle \tilde{D}_\sigma \rangle x^{\sigma(\infty)} y^{\sigma(0)} \\ &= (-A^3)^{-w(D)+sw(D)-sw(D)} \sum_{\sigma \in \mathfrak{S}(D)} \langle \tilde{D}_\sigma \rangle x^{\sigma(\infty)} y^{\sigma(0)} \\ &= (-A^3)^{-w(D)+sw(D)} \left((-A^3)^{-sw(\tilde{D})} \sum_{\sigma \in \mathfrak{S}(\tilde{D})} \langle \tilde{D}_\sigma \rangle x^{\sigma(\infty)} y^{\sigma(0)} \right) \\ &= (-A^3)^{sw(D)-w(D)} \ll \tilde{D} \gg. \end{aligned}$$

This completes the proof. \square

Theorem 5.4. (1) The normalized Kauffman bracket ideal is equal to the Kauffman bracket ideal I in Theorem 5.1 (1).

(2) For any oriented marked graph diagram D , the ideal coset $\ll D \gg_N^I = \ll D \gg_N + I$ is an invariant for the oriented surface-link in \mathbb{R}^4 or S^4 presented by D .

Proof. This is a direct consequence of Theorems 5.1 and 5.3. \square

5.3 Normal forms of Kauffman bracket ideal cosets

In this subsection, we show how to calculate the normal forms of the Kauffman bracket ideal cosets and the normalized Kauffman bracket ideal cosets implemented by the Groebner basis calculations on Maple 17 and compute the normal forms of surface-links in Yoshikawa's table [25] with $\text{ch-index} \leq 7$. For our purpose, we substitute B for A^{-1} and consider the ideal J of the polynomial ring $\mathbb{Q}[A, B, x, y]$ generated by the following four polynomials:

$$\begin{aligned} p_1(A, B, x, y) &= (-A^2 - B^2)x + y - 1, \\ p_2(A, B, x, y) &= x + (-A^2 - B^2)y - 1, \\ p_3(A, B, x, y) &= (A^4 + 1 + B^4)xy, \\ p_4(A, B, x, y) &= AB - 1. \end{aligned}$$

Using Groebner basis calculations on Maple 17, we see that

$$G = \{x - 1 + y, AB - 1, A^2 + B^2 + 1, B^3 + A + B\} \quad (5.17)$$

is a Groebner basis for the ideal $J = \langle p_1, p_2, p_3, p_4 \rangle$ with respect to the graded reverse lexicographic order "tdeg" (also called "grevlex" in the literature) in A, B, x, y . As a consequence of Theorems 4.7 and 5.4, we obtain the following theorem.

Theorem 5.5. Let D be an (resp. oriented) marked graph diagram and let \mathcal{L} be the (resp. oriented) surface-link presented by D . Let $\ll D \gg$ be the polynomial of D in (5.10) (resp. (5.16)). Then the normal form $\overline{\ll D \gg}$ (resp. $\overline{\ll D \gg}_N$) on division of $\ll D \gg$ (resp. $\ll D \gg_N$) by the Groebner basis G in (5.17) is an invariant of \mathcal{L} .

We denote the normal forms $\overline{\ll D \gg}$ and $\overline{\ll D \gg}_N$ in Theorem 5.5 by $\overline{\ll \mathcal{L} \gg}$ and $\overline{\ll \mathcal{L} \gg}_N$, respectively.

In the rest of this subsection, we give various examples which will be used in Section 6 again. First, we recall the normalized Kauffman bracket polynomials of the following oriented knots and links in 3-space which will be used in our discussion of examples:

$$\begin{aligned} \langle 2_1^2 \rangle_N &= \langle \text{diagram} \rangle_N = -A^{10} - A^2, \\ \langle 2_1^{2*} \rangle_N &= \langle \text{diagram} \rangle_N = -A^{-10} - A^{-2}, \\ \langle 3_1 \rangle_N &= \langle \text{diagram} \rangle_N = -A^{-16} + A^{-12} + A^{-4}, \\ \langle 4_1^2 \rangle_N &= \langle \text{diagram} \rangle_N = -A^{18} - A^{10} + A^6 - A^2, \\ \langle 4_1 \rangle_N &= \langle \text{diagram} \rangle_N = A^{-8} - A^{-4} + 1 - A^4 + A^8, \\ \langle 2_1^2 \# 3_1 \rangle_N &= \langle \text{diagram} \rangle_N = (-A^{10} - A^2)(-A^{-16} + A^{-12} + A^{-4}), \\ \langle 3_1^* \# 3_1 \rangle_N &= \langle \text{diagram} \rangle_N = (A^{16} - A^{12} - A^4)(A^{-16} - A^{-12} - A^{-4}). \end{aligned} \quad (5.18)$$

By (5) in Theorem 5.2, we obtain the following useful identities:

$$\begin{aligned} \ll \text{diag}_1 \gg &= A^{-8} \ll \text{diag}_2 \gg + (A^{-6} - A^{-2}) \ll \text{diag}_3 \gg, \\ \ll \text{diag}_4 \gg &= A^8 \ll \text{diag}_5 \gg + (A^6 - A^2) \ll \text{diag}_6 \gg. \end{aligned} \quad (5.19)$$

Example 5.6. Let 2_1^1 be the unoriented standard torus in Yoshikawa's table [25]. Then $sw(2_1^1) = 0$ and

$$\begin{aligned} \left[\left[\text{diag}_7 \right] \right]_{2_1^1} &= \langle \text{diag}_8 \rangle x^2 + \langle \text{diag}_9 \rangle xy + \langle \text{diag}_{10} \rangle yx + \langle \text{diag}_{11} \rangle y^2 \\ &= x^2 + 2(-A^2 - A^{-2})xy + y^2. \end{aligned}$$

Substituting B for A^{-1} , we get $\ll 2_1^1 \gg = (-A^3)^{-sw(2_1^1)}[[2_1^1]] = x^2 + 2(-A^2 - B^2)xy + y^2$. This yields the normal form $\ll 2_1^1 \gg = 1$. Further, for any orientation on 2_1^1 , $w(2_1^1) = sw(2_1^1) = 0$ and hence it follows from Theorem 5.3 that $\ll 2_1^1 \gg_N = \ll 2_1^1 \gg = 1$.

Example 5.7. Let 2_1^{-1} be the positive standard projective plane in Yoshikawa's table [25]. Then $sw(2_1^{-1}) = 0$ and

$$\left[\left[\text{diag}_{12} \right] \right]_{2_1^{-1}} = \langle \text{diag}_{13} \rangle x + \langle \text{diag}_{14} \rangle y = (-A^3)x + (-A^{-3})y.$$

Substituting B for A^{-1} , we have $\ll 2_1^{-1} \gg = (-A^3)^{-sw(2_1^{-1})}[[2_1^{-1}]] = -A^3x - B^3y$. This gives the normal form $\ll 2_1^{-1} \gg = A + B$. Let 2_1^{-1*} be the negative standard projective plane. Then we also have $\ll 2_1^{-1*} \gg = A + B$.

Example 5.8. Consider the 2 component surface-link $6_1^{0,1}$ in Yoshikawa's table [25] with the orientation indicated below. By Theorem 5.2 together with (5.19) and (5.18), we obtain

$$\begin{aligned} \ll \text{diag}_{15} \gg_N &= x^2 \ll \text{diag}_{16} \gg_N + xy \ll \text{diag}_{17} \gg_N + \\ &\quad yx \ll \text{diag}_{18} \gg_N + y^2 \ll \text{diag}_{19} \gg_N \\ &= \langle \text{diag}_{20} \rangle_N (x^2 + y^2) + xy \langle \text{diag}_{21} \rangle_N + yx \langle \text{diag}_{22} \rangle_N \\ &= (-A^{-2} - A^2)(x^2 + y^2) + (-A^{10} - A^2)(-A^{-10} - A^{-2})xy + (-A^{-2} - A^2)^2 xy \\ &= (-A^2 - A^{-2})(x^2 + y^2) + (A^4 + 4 + A^{-4} + A^{-8} + A^8)xy. \end{aligned}$$

Substituting B for A^{-1} , we get

$$\ll 6_1^{0,1} \gg_N = (-A^2 - B^2)(x^2 + y^2) + (A^4 + 4 + B^4 + B^8 + A^8)xy.$$

This yields the normal form $\overline{\ll 6_1^{0,1} \gg_N} = 1$. Further, forgetting the orientation on $6_1^{0,1}$, we see that $sw(\widetilde{6_1^{0,1}}) = w(6_1^{0,1}) = 0$ and hence it follows from Theorem 5.3 that $\overline{\ll 6_1^{0,1} \gg} = \overline{\ll 6_1^{0,1} \gg_N} = 1$.

Example 5.9. Consider the nonorientable two component surface-link $7_1^{0,-2}$ in Yoshikawa's table [25]. Then $sw(7_1^{0,-2}) = 1$ and

$$\begin{aligned} \ll 7_1^{0,-2} \gg &= (-A^3)^{-1} \left(x^2 \ll \text{diagram 1} \gg + xy \ll \text{diagram 2} \gg \right. \\ &\quad \left. + yx \ll \text{diagram 3} \gg + y^2 \ll \text{diagram 4} \gg \right) \\ &= x^2 \langle \bigcirc \bigcirc \rangle - A^{-3} xy \left(A \langle \bigcirc \bigcirc \rangle + A^{-1} \langle \text{diagram 5} \rangle \right) \\ &\quad + yx \langle \bigcirc \bigcirc \bigcirc \rangle + y^2 \langle \bigcirc \bigcirc \rangle \\ &= (-A^{-2} - A^2)(x^2 + y^2) + (-A^{-2} - A^2)^2 xy \\ &\quad - A^{-3} \left(A(-A^{-2} - A^2) + A^{-1}(-A^{-4} - A^4)^2 \right) xy \\ &= (-A^{-2} - A^2)(x^2 + y^2) + (3 - A^{-12})xy. \end{aligned}$$

Substituting B for A^{-1} , we get $\ll 7_1^{0,-2} \gg = (-A^2 - B^2)(x^2 + y^2) + (3 - B^{12})xy$. This gives $\ll 7_1^{0,-2} \gg = 1$.

A virtual marked graph diagram is a marked graph diagram possibly with virtual crossings indicated by small circles as usual in virtual link diagrams. In [11], L. H. Kauffman suggested the notion of isotopy of virtual surface-links in four space by means of virtual marked graph diagrams modulo a generalization of Yoshikawa moves on marked graph diagrams in purpose to investigate the relationships between this diagrammatic definition and more geometric approaches to virtual 2-knots. In [22], S. Nelson and P. Rivera introduced an isotopy invariant of virtual surface-links presented by virtual marked graph diagrams by using ribbon biquandles.

Note that the (normalized) Kauffman bracket polynomial is an invariant for virtual links [10]. This confirms that the construction of the ideal coset invariant associated with the (normalized) Kauffman bracket polynomial can be extended to (oriented) virtual surface-links presented by virtual marked graph diagrams and

consequently the ideal coset invariant associated with the (normalized) Kauffman bracket polynomial is also invariant for (oriented) virtual surface-links. In a separate paper [20], this extension will be dealt with full details in a more general setting. It is noted that the examples 5.6-5.9 above are implicit that the (normalized) Kauffman bracket ideal coset invariant seems to be almost trivial for surface-links and that this conjecture will be discussed in [20] in details. On the other hand, the following example 5.10 shows that the (normalized) Kauffman bracket ideal coset invariant is highly nontrivial for (oriented) virtual surface-links.

Example 5.10. Consider the oriented virtual S^2 -knot D below. Since

$$\begin{aligned} \langle \text{diagram} \rangle &= A \langle \text{diagram} \rangle + A^{-1} \langle \text{diagram} \rangle \\ &= A \langle \bigcirc \rangle + A^{-1} \left[A \langle \text{diagram} \rangle + A^{-1} \langle \text{diagram} \rangle \right] = A + 1 + A^{-2}, \end{aligned}$$

we have

$$\begin{aligned} \ll D \gg_N &= x^2 \ll \text{diagram} \gg_N + xy \ll \text{diagram} \gg_N + yx \ll \text{diagram} \gg_N + y^2 \ll \text{diagram} \gg_N \\ &= x^2 \langle \bigcirc \bigcirc \rangle_N + xy \langle \text{diagram} \rangle_N + yx \langle \bigcirc \bigcirc \bigcirc \rangle_N + y^2 \langle \bigcirc \bigcirc \rangle_N \\ &= (-A^{-2} - A^2)(x^2 + y^2) + (-A^{-2} - A^2)^2 xy + (A + 1 + A^{-2})xy \\ &= (-A^{-2} - A^2)(x^2 + y^2) + (A^4 + A + 3 + A^{-2} + A^{-4})xy. \end{aligned}$$

Substituting B for A^{-1} , we get

$$\ll D \gg_N = (-A^2 - B^2)(x^2 + y^2) + (A^4 + A + 3 + B^2 + B^4)xy.$$

This yields the normal form

$$\overline{\ll D \gg_N} = -B^2 y^2 - A y^2 + B^2 y + A y + 1.$$

In [20], it is seen that the normalized Kauffman bracket ideal coset invariant distills genuine oriented virtual marked graph diagrams from oriented virtual marked graph diagrams.

Question. Is there a regular or an ambient isotopy invariant $[\]$ for knots and links in 3-space such that the associated $[\]$ ideal is $\langle 0 \rangle$?

If such a link invariant $[\]$ exists, then it follows from Theorem 4.5 that it is naturally extended to a polynomial invariant for surface-links in 4-space.

6 A modification of the Kauffman bracket ideal

In this section, we consider a modification of the construction of the Kauffman bracket ideal in view of Corollary 4.6 which leads new invariants for (oriented) surface-links. We give the modification in the subsection 6.1 and compute the invariants for various surface-links in Yoshikawa's table [25] in the subsection 6.2.

6.1 A family of invariants for surface-links

Let

$$z(t) = \frac{1}{2} \left(\sqrt{3-t^{-1}} + \mathbf{i} \sqrt{1+t^{-1}} \right), \quad \bar{z}(t) = \frac{1}{2} \left(\sqrt{3-t^{-1}} - \mathbf{i} \sqrt{1+t^{-1}} \right),$$

where $t \neq 0$ and $\mathbf{i} = \sqrt{-1}$. We observe that $z(t)\bar{z}(t) = \frac{1}{4}(3-t^{-1}-\mathbf{i}^2(1+t^{-1})) = 1$ and so we have $\bar{z}(t) = z(t)^{-1}$. For any polynomial $f = f(A, A^{-1}, x, y) \in \mathbb{Z}[A, A^{-1}, x, y]$, we define ϕf by $\phi f = f(z(t), z(t)^{-1}, t, t) = f(z(t), \bar{z}(t), t, t)$. It is obvious that for every polynomial $f \in \mathbb{Z}[A, A^{-1}, x, y]$, the evaluation ϕf is expressed as the form:

$$\begin{aligned} \phi f = & a_0(t) + \mathbf{i}b_0(t) + \left(a_1(t) + \mathbf{i}b_1(t) \right) \sqrt{1+t^{-1}} \\ & + \left(a_2(t) + \mathbf{i}b_2(t) \right) \sqrt{3-t^{-1}} + \left(a_3(t) + \mathbf{i}b_3(t) \right) \sqrt{3+2t^{-1}-t^{-2}}, \end{aligned} \quad (6.20)$$

where $a_i(t), b_i(t) \in \mathbb{Z}[2^{-1}, t^{-1}, t] (0 \leq i \leq 3)$, namely, polynomials in variables 2^{-1} and $t^{\pm 1}$ with integral coefficients. Now we make the following definition:

Definition 6.1. Let D be an oriented marked graph diagram and let $\ll D \gg_N$ be the polynomial of D associated with the normalized Kauffman bracket polynomial $\langle \rangle_N$. We define $\mathbf{K}(D; t)_N$ (for short, $\mathbf{K}(D)_N$) by the formula:

$$\mathbf{K}(D)_N = \mathbf{K}(D; t)_N = \phi \ll D \gg_N = (-z(t)^3)^{-w(D)} [[\tilde{D}]](z(t), \bar{z}(t), t, t).$$

Lemma 6.2. Let D be an oriented marked graph diagram. Then $\mathbf{K}(D)_N$ is an invariant for all oriented Yoshikawa moves, except for the moves Γ_7 and Γ_8 .

Proof. The invariance for oriented Yoshikawa moves of type I is direct from Theorem 3.7. To prove the invariance of $\mathbf{K}(D)_N$ for the oriented Yoshikawa moves Γ_6 and Γ'_6 , we observe that $-z(t)^2 - z(t)^{-2} = -z(t)^2 - \bar{z}(t)^2 = t^{-1} - 1$. Note that $\delta = -A^2 - A^{-2}$ for the Kauffman bracket. In (4.6) and (4.7) in Lemma 4.1,

$$\begin{aligned} \phi(\delta x + y) &= (-z(t)^2 - z(t)^{-2})t + t = (t^{-1} - 1)t + t = 1, \\ \phi(x + \delta y) &= t + (-z(t)^2 - z(t)^{-2})t = t + (t^{-1} - 1)t = 1. \end{aligned}$$

This shows that $\mathbf{K}(D)_N$ is invariant under Γ_6 and Γ'_6 . This completes the proof. \square

Lemma 6.3. Let D be an oriented marked graph diagram and let D' be an oriented marked graph diagram obtained from D by applying a single Yoshikawa move Γ_7 or Γ_8 . Then

$$\mathbf{K}(D')_N = \mathbf{K}(D)_N + (2t - 1)\Psi(t), \quad (6.21)$$

where $\Psi(t)$ is of the form in (6.20).

Proof. Let D be an oriented marked graph diagram and let D' be the oriented marked graph diagram obtained from D by applying a single Yoshikawa move Γ_7 . Since $z(t)^4 + 1 + \bar{z}(t)^4 = (-z(t)^2 - \bar{z}(t)^2)^2 - 1 = (t^{-1} - 1)^2 - 1 = t^{-2} - 2t^{-1}$, it follows from (5.12) with the substitutions $A = z(t)$, $A^{-1} = \bar{z}(t)$, $x = y = t$ that

$$(\langle R(U) \rangle - \langle R^*(U) \rangle)t^2 = (1 - 2t)(\psi_3(U) - \psi_4(U)).$$

By Lemma 4.2 by taking the normalized Kauffman bracket, we have

$$\begin{aligned} \mathbf{K}(D')_N - \mathbf{K}(D)_N &= \phi(\ll D' \gg) - \phi(\ll D \gg) \\ &= (-z(t)^3)^{-w(D)} \sum_{k=1}^m \phi(\psi_k(A, A^{-1}, x, y))(1 - 2t)\phi(\psi_3(U_k) - \psi_4(U_k)) \\ &= (2t - 1)\Psi(t), \end{aligned}$$

where $\Psi(t) = (-z(t)^3)^{-w(D)} \sum_{k=1}^m \phi(\psi_k(A, A^{-1}, x, y))\phi(\psi_4(U_k) - \psi_3(U_k))$. Since $\psi_4(U_k) - \psi_3(U_k) \in \mathbb{Z}[A, A^{-1}, x, y]$ for all $k = 1, 2, \dots, m$, it is clear that $\Psi(t)$ has the form in (6.20).

Now let D be an oriented marked graph diagram and let D' be an oriented marked graph diagram obtained from D by applying a single oriented Yoshikawa move Γ_8 (cf. Figure 13). By a similar argument combined with Lemma 4.3 and (5.13), we obtain the identity (6.21). This completes the proof. \square

The following theorem 6.4, which gives a method of computing the polynomial $\mathbf{K}(D)_N$ recursively for a given oriented marked graph diagram D .

Theorem 6.4. Let D be an oriented marked graph diagram.

- (1) $\mathbf{K}(\bigcirc)_N = 1$.
- (2) If D and D' are two oriented marked graph diagrams related by a finite sequence of oriented Yoshikawa moves generated by the moves $\Gamma_1, \Gamma'_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma'_4, \Gamma_5, \Gamma_6$ and Γ'_6 , then $\mathbf{K}(D)_N = \mathbf{K}(D')_N$.
- (3) $\mathbf{K}(D \sqcup \bigcirc)_N = (t^{-1} - 1)\mathbf{K}(D)_N$.
- (4) $\mathbf{K}\left(\begin{array}{c} \nearrow \quad \nwarrow \\ \nwarrow \quad \nearrow \end{array}\right)_N = t \left[\mathbf{K}\left(\begin{array}{c} \nearrow \quad \nearrow \\ \nwarrow \quad \nwarrow \end{array}\right)_N + \mathbf{K}\left(\begin{array}{c} \nearrow \quad \nwarrow \\ \nwarrow \quad \nearrow \end{array}\right)_N \right]$.
- (5) $\lambda(t)\mathbf{K}\left(\begin{array}{c} \nearrow \quad \nearrow \\ \nwarrow \quad \nwarrow \end{array}\right)_N - \bar{\lambda}(t)\mathbf{K}\left(\begin{array}{c} \nearrow \quad \nwarrow \\ \nwarrow \quad \nearrow \end{array}\right)_N = \mathbf{i}\sqrt{1+t^{-1}}\sqrt{3-t^{-1}} \mathbf{K}\left(\begin{array}{c} \nearrow \quad \nwarrow \\ \nwarrow \quad \nearrow \end{array}\right)_N$,
where D_+, D_-, D_0 are three identical oriented link diagrams except the parts indicated and $\lambda(t) = 2^{-1}t^{-2}\left((t^2 + 2t - 1) - \mathbf{i}(t^2 - t)\sqrt{1+t^{-1}}\sqrt{3-t^{-1}}\right)$.

Proof. From Theorem 5.2 with the substitutions $A = z(t)$, $A^{-1} = \bar{z}(t)$, $x = y = t$ and Lemma 6.2, we obtain the assertions (1), (2), (3) and (4) directly. Observe that

$$\begin{aligned} z(t)^4 &= -2^{-1}t^{-2}\left((t^2 + 2t - 1) - (t^2 - t)\mathbf{i}\sqrt{1+t^{-1}}\sqrt{3-t^{-1}}\right), \\ \bar{z}(t)^4 &= -2^{-1}t^{-2}\left(t^2 + 2t - 1 + (t^2 - t)\mathbf{i}\sqrt{1+t^{-1}}\sqrt{3-t^{-1}}\right), \\ \bar{z}(t)^2 - z(t)^2 &= -\mathbf{i}\sqrt{1+t^{-1}}\sqrt{3-t^{-1}}. \end{aligned}$$

Hence the assertions (5) follows immediately from the skein relation in (5) in Theorem 5.2. This completes the proof. \square

Let \mathbb{C} denote the field of all complex numbers. For each integer $n \geq 2$, let $\phi_n : \mathbb{Z}[A, A^{-1}, x, y] \rightarrow \mathbb{C}$ be the function defined by

$$\phi_n(f) = \phi f|_{t=n} = f(z(n), \bar{z}(n), n, n)$$

for each polynomial $f = f(A, A^{-1}, x, y) \in \mathbb{Z}[A, A^{-1}, x, y]$. Then ϕ_n is a ring homomorphism and the image of ϕ_n is a subring of \mathbb{C} . From (6.20), we see that $\phi_n(f)$ is expressed as a complex number of the form:

$$\begin{aligned} \phi_n(f) = & a_0(n) + \mathbf{i}b_0(n) + \left(a_1(n) + \mathbf{i}b_1(n)\right)\sqrt{1+n^{-1}} + \left(a_2(n) + \mathbf{i}b_2(n)\right)\sqrt{3-n^{-1}} \\ & + \left(a_3(n) + \mathbf{i}b_3(n)\right)\sqrt{1+n^{-1}}\sqrt{3-n^{-1}}, \end{aligned} \quad (6.22)$$

where $a_i(n), b_i(n) \in \mathbb{Z}[2^{-1}, n^{-1}]$ ($0 \leq i \leq 3$), namely, polynomials in variables 2^{-1} and n^{-1} with integral coefficients. Let

$$\Lambda_n = \{\phi_n(f) \mid f \in \mathbb{Z}[A, A^{-1}, x, y]\}.$$

Now let $\mathbb{Z}_{2n-1} = \{[0], [1], [2], \dots, [2n-1]\}$ denote the factor ring $\mathbb{Z}/(2n-1)\mathbb{Z}$ of the ring \mathbb{Z} of integers modulo $2n-1$ and let $\bar{\phi}_n(f)$ denote $\phi_n(f)$ reducing integral coefficients of each terms $a_i(n)$ and $b_i(n)$ ($0 \leq i \leq 3$) modulo $2n-1$ (hence the reduced coefficients are all in \mathbb{Z}_{2n-1}). Then we have the following:

Theorem 6.5. Let \mathcal{L} be an oriented surface-link and let D be an oriented marked graph diagram presenting \mathcal{L} . Then for each integer $n \geq 2$, $\mathbf{K}_{2n-1}(D)_N = \overline{\mathbf{K}(D; n)}_N$ is an invariant of \mathcal{L} , and denoted by $\mathbf{K}_{2n-1}(\mathcal{L})_N$.

Proof. By Lemma 6.2, $\mathbf{K}(D; n)$ is invariant under oriented Yoshikawa moves of type I and the moves Γ_6 and Γ'_6 and so does $\mathbf{K}_{2n-1}(D) = \overline{\mathbf{K}(D; n)}_N$ for each integer $n \geq 2$.

Let D be an oriented marked graph diagram and let D' be an oriented marked graph diagram obtained from D by applying a single oriented Yoshikawa move Γ_7 or Γ_8 . By Lemma 6.3, we see that for each integer $n \geq 2$,

$$\mathbf{K}(D; n)_N = \mathbf{K}(D'; n)_N + (2n-1)\Psi(n),$$

where $\Psi(n) \in \Lambda_n$. Hence we have

$$\begin{aligned} \mathbf{K}_{2n-1}(D)_N &= \overline{\mathbf{K}(D; n)}_N = \overline{\mathbf{K}(D'; n)_N + (2n-1)\Psi(n)} \\ &= \overline{\mathbf{K}(D'; n)}_N = \mathbf{K}_{2n-1}(D')_N. \end{aligned}$$

This completes the proof. \square

Definition 6.6. Let D be a marked graph diagram and let $\ll D \gg$ be the polynomial of D associated with the Kauffman bracket polynomial. Define $\mathbf{K}(D)$ by

$$\mathbf{K}(D) = \mathbf{K}(D; t) = \phi \ll D \gg = (-z(t)^3)^{-sw(D)}[[D]](z(t), \bar{z}(t), t, t).$$

By parallel argument of the proof forgetting orientation, we obtain the corresponding Lemmas 6.2 and 6.3 for $\mathbf{K}(D)$ and consequently we obtain the following:

Theorem 6.7. Let \mathcal{L} be a surface-link and let D be a marked graph diagram presenting \mathcal{L} . Then for each integer $n \geq 2$, $\mathbf{K}_{2n-1}(D) = \overline{\mathbf{K}(D; n)}$ is an invariant of \mathcal{L} , and denoted by $\mathbf{K}_{2n-1}(\mathcal{L})$.

Remark 6.8. For each pair $\epsilon = (\epsilon_1, \epsilon_2) \in \{(1, -1), (-1, 1), (-1, -1)\}$, let

$$z_\epsilon(t) = \frac{1}{2}(\epsilon_1 \sqrt{3-t^{-1}} + \epsilon_2 \mathbf{i} \sqrt{1+t^{-1}}), \quad \overline{z}_\epsilon(t) = \frac{1}{2}(\epsilon_1 \sqrt{3-t^{-1}} - \epsilon_2 \mathbf{i} \sqrt{1+t^{-1}}),$$

where $t \neq 0$. It is easily seen that $z_\epsilon(t)\overline{z}_\epsilon(t) = 1$ and $-z_\epsilon(t)^2 - z_\epsilon(t)^{-2} = -z_\epsilon(t)^2 - \overline{z}_\epsilon(t)^2 = t^{-1} - 1$. In the proof of Lemma 6.2 with $\delta = -A^2 - A^{-2}$, it is checked that

$$\begin{aligned} \phi(\delta x + y) &= (-z_\epsilon(t)^2 - z_\epsilon(t)^{-2})t + t = (t^{-1} - 1)t + t = 1, \\ \phi(x + \delta y) &= t + (-z_\epsilon(t)^2 - z_\epsilon(t)^{-2})t = t + (t^{-1} - 1)t = 1. \end{aligned}$$

In the proof of Lemma 6.3, it is checked that $z_\epsilon(t)^4 + 1 + \overline{z}_\epsilon(t)^4 = (-z_\epsilon(t)^2 - \overline{z}_\epsilon(t)^2)^2 - 1 = (t^{-1} - 1)^2 - 1 = t^{-2} - 2t^{-1}$ and it also follows from (5.12) with the substitution $A = z_\epsilon(t)$, $A^{-1} = z_\epsilon(t)^{-1}$, $x = y = t$ that $(\langle R(U) \rangle - \langle R^*(U) \rangle)t^2 = (1 - 2t)(\psi_3(U) - \psi_4(U))$. Now we define

$$\begin{aligned} \mathbf{K}_\epsilon(D; t) &= \phi_\epsilon \ll D \gg = (-z_\epsilon(t)^3)^{-sw(D)}[[D]](z_\epsilon(t), \overline{z}_\epsilon(t), t, t), \\ \mathbf{K}_\epsilon(D; t)_N &= \phi_\epsilon \ll D \gg_N = (-z_\epsilon(t)^3)^{-w(D)}[[\tilde{D}]](z_\epsilon(t), \overline{z}_\epsilon(t), t, t), \end{aligned}$$

where $\phi_\epsilon f = f(z_\epsilon(t), \overline{z}_\epsilon(t), t, t)$ for $f \in \mathbb{Z}[A, A^{-1}, x, y]$. By the same argument of the proof of Theorem 6.5 with $\phi = \phi_\epsilon$, we obtain that for each integer $n \geq 2$, $\mathbf{K}_{2n-1}^\epsilon(D) = \overline{\mathbf{K}_\epsilon(D; n)}$ (resp. $\mathbf{K}_{2n-1}^\epsilon(D)_N = \overline{\mathbf{K}_\epsilon(D; n)_N}$) is also an invariant of an (resp. oriented) surface-link \mathcal{L} presented by an (resp. oriented) marked graph diagram D .

6.2 Examples

In this subsection, we calculate the invariants $\mathbf{K}_{2n-1}(\mathcal{L})$ and $\mathbf{K}_{2n-1}(\mathcal{L})_N$ for various surface-links \mathcal{L} .

Example 6.9. Let 2_1^1 be the standard torus of genus one with the orientation in Example 5.6, where it is seen that $\ll \tilde{2}_1^1 \gg = \ll 2_1^1 \gg_N = x^2 + 2(-A^2 - A^{-2})xy + y^2$. This gives that for each integer $n \geq 2$, $\mathbf{K}(\tilde{2}_1^1; n) = \mathbf{K}(2_1^1; n)_N = n^2 + 2(-z(n)^2 - \overline{z}(n)^2)n^2 + n^2 = 2n$. Therefore

$$\mathbf{K}_{2n-1}(\tilde{2}_1^1) = \mathbf{K}_{2n-1}(2_1^1)_N = \overline{2n} = [1] \quad (n \geq 2).$$

Example 6.10. Let 2_1^{-1} be the positive standard projective plane. Then it is seen from Example 5.7 that $\ll 2_1^{-1} \gg = (-A^3)x + (-A^{-3})y$. This gives that for each integer $n \geq 2$, $\mathbf{K}(2_1^{-1}; n) = (-z(n)^3)n + (-\overline{z}(n)^3)n = n^{-\frac{1}{2}}\sqrt{3n-1}$. Therefore

$$\mathbf{K}_{2n-1}(2_1^{-1}) = [1]n^{-\frac{1}{2}}\sqrt{3n-1} \quad (n \geq 2).$$

Similarly, let 2_1^{-1*} be the negative standard projective plane. Then for each $n \geq 2$, we have $\mathbf{K}_{2n-1}(2_1^{-1*}) = \mathbf{K}_{2n-1}(2_1^{-1}) = [1]n^{-\frac{1}{2}}\sqrt{3n-1}$.

Example 6.11. Let $6_1^{0,1}$ be the two component orientable surface-link with the orientation in Example 5.8, where we have $\ll 6_1^{0,1} \gg_N = (-A^2 - A^{-2})(x^2 + y^2) + (A^4 + 4 + A^{-4} + A^{-8} + A^8)xy$. This gives $\mathbf{K}(6_1^{0,1}; t)_N = \ll 6_1^{0,1} \gg_N (z(t), \bar{z}(t), t, t) = (4t^3 + 3t^2 - 4t + 1)t^{-2}$. Hence for each integer $n \geq 2$,

$$\mathbf{K}_{2n-1}(6_1^{0,1})_N = \overline{\mathbf{K}(6_1^{0,1}; n)_N} = [4n^3 + 3n^2 - 4n + 1]n^{-2}.$$

Further, forgetting the orientation on $6_1^{0,1}$, we see from Example 5.8 that $\ll 6_1^{0,1} \gg_N = \ll \widehat{6_1^{0,1}} \gg$ and therefore $\mathbf{K}_{2n-1}(\widehat{6_1^{0,1}}) = \mathbf{K}_{2n-1}(6_1^{0,1})_N$ for all $n \geq 2$.

Example 6.12. Let $7_1^{0,-2}$ be the two component nonorientable surface-link in Example 5.9, where we get $\ll 7_1^{0,-2} \gg = (-A^{-2} - A^2)(x^2 + y^2) + (3 - A^{-12})xy$. This gives $\mathbf{K}(7_1^{0,-2}; t) = \ll 7_1^{0,-2} \gg (z(t), \bar{z}(t), t, t) = (4t^5 + 12t^4 - 4t^3 - 9t^2 + 6t - 1)2^{-1}t^{-4} + (4t^4 - 2t^3 - 6t^2 + 5t - 1)2^{-1}t^{-4}\mathbf{i}\sqrt{(t+1)(3t-1)}$. Hence for each integer $n \geq 2$,

$$\begin{aligned} \mathbf{K}_{2n-1}(7_1^{0,-2}) = \overline{\mathbf{K}(7_1^{0,-2}; n)} &= [4n^5 + 12n^4 - 4n^3 - 9n^2 + 6n - 1]2^{-1}n^{-4} \\ &+ [4n^4 - 2n^3 - 6n^2 + 5n - 1]2^{-1}n^{-4}\mathbf{i}\sqrt{(n+1)(3n-1)}. \end{aligned}$$

Example 6.13. Let 8_1 be the spun 2-knot of the trefoil in Yoshikawa's table (cf. [12, 25]) with the orientation indicated below. By Theorem 5.2 together with (5.18) and (5.19), we have

$$\begin{aligned} \ll 8_1 \gg_N &= x^2 \ll \text{diagram 1} \gg_N + xy \ll \text{diagram 2} \gg_N + \\ &+ yx \ll \text{diagram 3} \gg_N + y^2 \ll \text{diagram 4} \gg_N \\ &= \langle \bigcirc \bigcirc \rangle_N (x^2 + y^2) + xy \langle \text{diagram 5} \rangle_N + yx \langle \bigcirc \bigcirc \bigcirc \rangle_N \\ &= (-A^{-2} - A^2)(x^2 + y^2) + (-A^{-2} - A^2)^2 xy + \\ &\quad (-A^{16} + A^{12} + A^4)(-A^{-16} + A^{-12} + A^{-4})xy \\ &= (-A^2 - A^{-2})(x^2 + y^2) + (5 - A^{12} - A^{-12} + A^8 + A^{-8})xy. \end{aligned}$$

This yields $\mathbf{K}(8_1; t)_N = \ll 8_1 \gg_N (z(t), \bar{z}(t), t, t) = (6t^5 + 14t^4 - 8t^3 - 8t^2 + 6t - 1)t^{-4}$. Hence for each integer $n \geq 2$,

$$\mathbf{K}_{2n-1}(8_1)_N = \overline{\mathbf{K}(8_1; n)_N} = [6n^5 + 14n^4 - 8n^3 - 8n^2 + 6n - 1]n^{-4}.$$

Further, forgetting the orientation on 8_1 , we see that $\ll 8_1 \gg_N = \ll \widetilde{8_1} \gg$ and therefore $\mathbf{K}_{2n-1}(\widetilde{8_1}) = \mathbf{K}_{2n-1}(8_1)_N$ for all $n \geq 2$.

Example 6.14. Let 9_1 be the ribbon 2-knot associated with the knot 6_1 in Yoshikawa's table with the orientation indicated below. By Theorem 5.2 together with (5.18) and (5.19), we obtain

$$\begin{aligned}
& \ll \text{Diagram } 9_1 \gg_N = x^2 \ll \text{Diagram } 1 \gg_N + xy \ll \text{Diagram } 2 \gg_N \\
& + yx \ll \text{Diagram } 3 \gg_N + y^2 \ll \text{Diagram } 4 \gg_N \\
& = x^2 \langle \bigcirc \bigcirc \rangle_N + xy \left(A^{-8} \langle \text{Diagram } 5 \rangle_N + (A^{-6} - A^{-2}) \langle \text{Diagram } 6 \rangle_N \right) \\
& + yx \langle \bigcirc \bigcirc \bigcirc \rangle_N + y^2 \langle \bigcirc \bigcirc \rangle_N \\
& = (-A^{-2} - A^2)(x^2 + y^2) + (-A^{-2} - A^2)^2 xy \\
& + \left(A^{-8}(A^{-8} - A^{-4} + 1 - A^4 + A^8) + (A^{-6} - A^{-2})(-A^{10} - A^2) \right) xy \\
& = (-A^{-2} - A^2)(x^2 + y^2) + (4 - A^{-4} + A^{-16} - A^{-12} + A^{-8} + A^8)xy.
\end{aligned}$$

This gives $\mathbf{K}(9_1; t)_N = \ll 9_1 \gg_N (z(t), \bar{z}(t), t, t) = (6t^7 + 27t^6 + 12t^5 - 37t^4 - 2t^3 + 19t^2 - 8t + 1)2^{-1}t^{-6} + (6t^6 + 7t^5 - 21t^4 + 14t^2 - 7t + 1)2^{-1}t^{-6}\mathbf{i}\sqrt{(t+1)(3t-1)}$. Hence for each integer $n \geq 2$,

$$\begin{aligned}
\mathbf{K}_{2n-1}(9_1)_N &= \overline{\mathbf{K}(9_1; n)_N} \\
&= [6n^7 + 27n^6 + 12n^5 - 37n^4 - 2n^3 + 19n^2 - 8n + 1]2^{-1}n^{-6} \\
&+ [6n^6 + 7n^5 - 21n^4 + 14n^2 - 7n + 1]2^{-1}n^{-6}\mathbf{i}\sqrt{(n+1)(3n-1)}.
\end{aligned}$$

Further, forgetting the orientation on 9_1 , we see that $\ll 9_1 \gg_N = \ll \tilde{9}_1 \gg$ and therefore $\mathbf{K}_{2n-1}(\tilde{9}_1) = \mathbf{K}_{2n-1}(9_1)_N$ for all $n \geq 2$.

Example 6.15. Let 10_2 be the 2-twist spun 2-knot of the trefoil in Yoshikawa's table with the orientation indicated below. By Theorem 5.2 together with (5.18) and (5.19), we obtain

$$\begin{aligned}
& \ll \text{Diagram } 10_2 \gg_N = x^2 \ll \text{Diagram } 1 \gg_N + xy \ll \text{Diagram } 2 \gg_N \\
& + yx \ll \text{Diagram } 3 \gg_N + y^2 \ll \text{Diagram } 4 \gg_N
\end{aligned}$$

$$\begin{aligned}
 &= x^2 \langle \bigcirc \bigcirc \rangle_N + xy \left(A^{-8} \langle \bigcirc \rangle_N + (A^{-6} - A^{-2}) \langle \text{diagram} \rangle_N \right) \\
 &\quad + xy \left(A^{-8} \langle \text{diagram} \rangle_N + (A^{-6} - A^{-2}) \langle \text{diagram} \rangle_N \right) \\
 &\quad + y^2 \left(A^{-8} \langle \text{diagram} \rangle_N + (A^{-6} - A^{-2}) \langle \bigcirc \rangle_N \right) \\
 &= x^2 (-A^{-2} - A^2) + xy \left(A^{-8} + (A^{-6} - A^{-2})(-A^{18} - A^{10} + A^6 - A^2) \right) \\
 &\quad + xy \left(A^{-8}(A^{-8} - A^{-4} + 1 - A^4 + A^8) + (A^{-6} - A^{-2})(-A^{10} - A^2) \right) \\
 &\quad + y^2 \left(A^{-8}(-A^{10} - A^2) + (A^{-6} - A^{-2}) \right) \\
 &= (-A^{-2} - A^2)(x^2 + y^2) + \\
 &\quad (A^{-16} - A^{-12} + 2A^{-8} - 3A^{-4} + 4 - 3A^4 + 2A^8 - A^{12} + A^{16})xy.
 \end{aligned}$$

This gives $\mathbf{K}(10_2; t)_N = \ll 10_2 \gg_N (z(t), \bar{z}(t), t, t) = (8t^7 + 25t^6 + 12t^5 - 37t^4 - 2t^3 + 19t^2 - 8t + 1)t^{-6}$. Hence for each integer $n \geq 2$,

$$\mathbf{K}_{2n-1}(10_2)_N = \overline{\mathbf{K}(10_2; n)}_N = [8n^7 + 25n^6 + 12n^5 - 37n^4 - 2n^3 + 19n^2 - 8n + 1]n^{-6}.$$

Further, forgetting the orientation on 10_2 , we see that $\ll 10_2 \gg_N = \ll \widetilde{10_2} \gg$ and therefore $\mathbf{K}_{2n-1}(\widetilde{10_2}) = \mathbf{K}_{2n-1}(10_2)_N$ for all $n \geq 2$.

Example 6.16. Consider the two component nonorientable surface-link $8_1^{-1,-1}$ in Yoshikawa's table. Observe that $sw(8_1^{-1,-1}) = 0$. By (5.10), (5.19) and Definition 6.6, we obtain

$$\begin{aligned}
 &\ll \text{diagram} \gg = [[\text{diagram}]] = x^2 [[\text{diagram}]] + \\
 &\quad xy [[\text{diagram}]] + yx [[\text{diagram}]] + y^2 [[\text{diagram}]] \\
 &= x^2 \langle \bigcirc \bigcirc \rangle + xy \left(A^{-1}(-A^{-3}) \langle \bigcirc \rangle + A(-A^3)^2 \langle \widetilde{3_1} \rangle \right) \\
 &\quad + xy \left(A(-A^{-3}) \langle \bigcirc \bigcirc \rangle + A^{-1}(-A^{-3}) \langle \text{diagram} \# \text{diagram} \rangle \right) \\
 &\quad + y^2 \left(A(-A^{-3}) \langle \bigcirc \rangle + A^{-1}(-A^3) \langle \bigcirc \rangle \right) \\
 &= x^2 (-A^{-2} - A^2) + xy \left(-A^{-4} + A^7(-A^5 - A^{-3} + A^{-7}) \right) \\
 &\quad + xy \left(-A^{-2}(-A^{-2} - A^2) + (-A^{-4})(-A^4 - A^{-4})^2 \right) + y^2 (-A^{-2} - A^2) \\
 &= (-A^{-2} - A^2)(x^2 + y^2) + (-A^{-12} - 2A^{-4} + 2 - 2A^4 - A^{12})xy.
 \end{aligned}$$

This gives $\mathbf{K}(8_1^{-1,-1}; t) = \ll 8_1^{-1,-1} \gg (z(t), \bar{z}(t), t, t) = (6t^5 + 10t^4 - 4t^3 - 9t^2 + 6t - 1)t^{-4}$. Hence for each integer $n \geq 2$,

$$\mathbf{K}_{2n-1}(8_1^{-1,-1}) = \overline{\mathbf{K}(8_1^{-1,-1}; n)} = [6n^5 + 10n^4 - 4n^3 - 9n^2 + 6n - 1]n^{-4}.$$

Example 6.17. Consider the two torus component surface-link $8_1^{1,1}$ in Yoshikawa's table with the orientation indicated below. By Theorem 6.4 together with (5.18) and (5.19), we obtain

$$\begin{aligned}
& \mathbf{K} \left(\text{Diagram of } 8_1^{1,1} \right)_N = t^2 \mathbf{K} \left(\text{Diagram 1} \right)_N + t^2 \mathbf{K} \left(\text{Diagram 2} \right)_N + \\
& t^2 \mathbf{K} \left(\text{Diagram 3} \right)_N + t^2 \mathbf{K} \left(\text{Diagram 4} \right)_N = t^2 \mathbf{K} \left(\text{Diagram 5} \right)_N + \\
& t^2 \mathbf{K} \left(\text{Diagram 6} \right)_N + t^2 \mathbf{K} \left(\text{Diagram 7} \right)_N + t^2 \mathbf{K} \left(\text{Diagram 8} \right)_N \\
& = t^2 \mathbf{K}(O^2)_N + t^3 \mathbf{K}(2_1^2 \# 2_1^{2*})_N + t^3 \mathbf{K}(O^2)_N + t^3 \mathbf{K}(2_1^2 \# 2_1^{2*})_N + t^3 \mathbf{K}(O^2)_N \\
& \quad + t^4 \mathbf{K}(2_1^2 \sqcup 2_1^{2*})_N + t^4 \mathbf{K}(2_1^2 \# 2_1^{2*})_N + t^4 \mathbf{K}(2_1^2 \# 2_1^{2*})_N + t^4 \mathbf{K}(O^2)_N \\
& = (t^2 + 2t^3 + t^4)(t^{-1} - 1) + \left(2(t^4 + t^3) + t^4(t^{-1} - 1) \right) (z(t)^{10} + z(t)^2)(\bar{z}(t)^{10} + \bar{z}(t)^2) \\
& = (6t^4 + 15t^3 + 3t^2 - 11t + 3)t^{-1}.
\end{aligned}$$

Hence for each integer $n \geq 2$,

$$\mathbf{K}_{2n-1}(8_1^{1,1})_N = \overline{\mathbf{K}(8_2^{1,1}; n)_N} = [6n^4 + 15n^3 + 3n^2 - 11n + 3]n^{-1}.$$

Further, forgetting the orientation on $8_1^{1,1}$, we see that $\ll 8_1^{1,1} \gg_N = \ll \widetilde{8_1^{1,1}} \gg$ and therefore $\mathbf{K}_{2n-1}(\widetilde{8_1^{1,1}}) = \mathbf{K}_{2n-1}(8_1^{1,1})_N$ for all $n \geq 2$.

Example 6.18. Let 10_1^1 be the spun torus of the trefoil in Yoshikawa's table with the orientation indicated below. By Theorem 6.4 together with (5.18) and (5.19), we obtain

$$\begin{aligned}
& \mathbf{K} \left(\text{Diagram of } 10_1^1 \right)_N = t^2 \mathbf{K} \left(\text{Diagram 1} \right)_N + t^2 \mathbf{K} \left(\text{Diagram 2} \right)_N + \\
& t^2 \mathbf{K} \left(\text{Diagram 3} \right)_N + t^2 \mathbf{K} \left(\text{Diagram 4} \right)_N = t^2 \mathbf{K} \left(\text{Diagram 5} \right)_N + \\
& t^2 \mathbf{K} \left(\text{Diagram 6} \right)_N + t^2 \mathbf{K} \left(\text{Diagram 7} \right)_N + t^2 \mathbf{K} \left(\text{Diagram 8} \right)_N
\end{aligned}$$

$$\begin{aligned}
 &= t^2 \mathbf{K}(O^2)_N + t^3 \mathbf{K}(3_1 \# 3_1^*)_N + t^3 \mathbf{K}(O^2)_N + t^3 \mathbf{K}(3_1 \# 3_1^*)_N + t^3 \mathbf{K}(O^2)_N \\
 &\quad + t^4 \mathbf{K}(3_1 \sqcup 3_1^*)_N + t^4 \mathbf{K}(3_1 \# 3_1^*)_N + t^4 \mathbf{K}(3_1 \# 3_1^*)_N + t^4 \mathbf{K}(O^2)_N \\
 &= (t^2 + 2t^3 + t^4)(t^{-1} - 1) + \left(2(t^4 + t^3) + t^4(t^{-1} - 1) \right) \\
 &\quad (-z(t)^{16} + z(t)^{12} + z(t)^4)(-\bar{z}(t)^{16} + \bar{z}(t)^{12} + \bar{z}(t)^4) \\
 &= (8t^6 + 32t^5 + 32t^4 - 32t^3 - 18t^2 + 17t - 3)t^{-3}.
 \end{aligned}$$

Hence for each integer $n \geq 2$,

$$\mathbf{K}_{2n-1}(10_1^1)_N = \overline{\mathbf{K}(10_1^1; n)_N} = [8n^6 + 32n^5 + 32n^4 - 32n^3 - 18n^2 + 17n - 3]n^{-3}.$$

Further, forgetting the orientation on 10_1^1 , we see that $\ll 10_1^1 \gg_N = \ll \widetilde{10_1^1} \gg$ and therefore $\mathbf{K}_{2n-1}(\widetilde{10_1^1}) = \mathbf{K}_{2n-1}(10_1^1)_N$ for all $n \geq 2$.

As a summary of the above discussion of examples, we give the following Table I of the first four invariants $\mathbf{K}_{2n-1}(\mathcal{L})_N (n = 2, 3, 4, 5)$ of all unoriented surface-links \mathcal{L} in Yoshikawa's table with ch-index ≤ 8 and three more 2-knots $9_1, 10_2$ and 10_1^1 .

Surface-link \mathcal{L}	$\mathbf{K}_3(\mathcal{L})$	$\mathbf{K}_5(\mathcal{L})$	$\mathbf{K}_7(\mathcal{L})$	$\mathbf{K}_9(\mathcal{L})$
0_1	[1]	[1]	[1]	[1]
2_1^1	[1]	[1]	[1]	[1]
2_1^{-1}	$[1]2^{-\frac{1}{2}}\sqrt{5}$	$[1]3^{-\frac{1}{2}}\sqrt{8}$	$[1]4^{-\frac{1}{2}}\sqrt{11}$	$[1]5^{-\frac{1}{2}}\sqrt{14}$
$6_1^{0,1}$	$[1]2^{-2}$	$[4]3^{-2}$	$[2]4^{-2}$	$[7]5^{-2}$
$7_1^{0,-2}$	$[2]2^{-5}$	$[1]2^{-1}3^{-4}$	$[1]2^{-1}4^{-4}$	$[4]2^{-1}5^{-4}$
8_1	$[1]2^{-4}$	$[1]3^{-4}$	$[4]4^{-4}$	$[4]5^{-4}$
$8_1^{1,1}$	$[2]2^{-1}$	[1]	$[4]4^{-1}$	$[5]5^{-1}$
$8_1^{-1,-1}$	$[1]2^{-4}$	$[1]3^{-4}$	$[4]4^{-4}$	$[4]5^{-4}$
9_1	$[2]2^{-7}$	$[4]3^{-7}$	$[2]4^{-7}$	$[1]5^{-7}$
10_2	$[1]2^{-6}$	$[4]3^{-6}$	$[1]4^{-6}$	$[1]5^{-6}$
10_1^1	$[2]2^{-3}$	$[4]3^{-2}$	$[1]4^{-3}$	$[8]5^{-3}$

Table I. The invariants $\mathbf{K}_{2n-1}(\mathcal{L}) (n = 2, 3, 4, 5)$ of surface-links \mathcal{L}

We close this subsection with the following remarks come from Table I.

- Remark 6.19.** (1) The invariants $\mathbf{K}_{2n-1}(\mathcal{L}) (n = 2, 3, 4, 5)$ distinguish the spun 2-knot 8_1 , the 2-twist spun 2-knot 10_2 and the spun torus 10_1^1 of the trefoil knot 3_1 .
- (2) The invariants $\mathbf{K}_{2n-1}(\mathcal{L}) (n = 2, 3, 4, 5)$ distinguish two nonorientable surface-links $7_1^{0,-2}$ and $8_1^{-1,-1}$ of the same nonorientable total genus.
- (3) The invariants $\mathbf{K}_{2n-1}(\mathcal{L}) (n = 2, 3, 4, 5)$ distinguish two component orientable surface-links $6_1^{0,1}$ and $8_1^{1,1}$, which have the same surface-link group $\mathbb{Z} \oplus \mathbb{Z}$ and so have the same Alexander ideal.
- (4) The invariants $\mathbf{K}_{2n-1}(\mathcal{L}) (n = 2, 3, 4, 5)$ distinguish the ribbon 2-knot 9_1 associated with the knot 6_1 from the spun 2-knots $8_1, 10_2$ and 10_2 .

- (5)) The 2-knot 8_1 and the nonorientable surface-link $8_1^{-1,-1}$ have the same invariant $\mathbf{K}_{2n-1}(\mathcal{L})(n = 2, 3, 4, 5)$.
- (6) $\mathbf{K}_5(6_1^{0,1})_N = [4]3^{-2} = \mathbf{K}_5(10_1^1)_N = [4]3^{-2}$. But, the other three invariants $\mathbf{K}_3(\mathcal{L}), \mathbf{K}_7(\mathcal{L})$ and $\mathbf{K}_9(\mathcal{L})$ distinguish two surface-links $6_1^{0,1}$ and 10_1^1 .

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